Embedding types and canonical affine maps between Bruhat-Tits buildings of classical groups

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Abstract

Introduction

This thesis is devoted to the description and characterisation of affine maps between enlarged Bruhat-Tits buildings of certain reductive groups over non-Archimedean local fields. More precisely we consider subgroups of classical groups which arise as the centraliser of a rational Lie algebra element which generate a semisimple algebra over the base field, and we study affine embeddings of the corresponding Bruhat-Tits buildings. Our approach is based on the fact that the building can be described in terms of lattice functions.

In part two we consider unit groups of local central simple algebras or in other words general linear groups with coefficients in a local division algebra under no assumption on the characteristic. Here we use the affine embeddings of Bruhat-Tits buildings of centraliser subgroups in order to recover the embedding data from the work of Broussous and Grabitz. Embedding data play an important role in the construction of simple types. It is to underline the usefulness of such affine embeddings and that we may expect further results in the future.

Part 1

For the construction of types for p-adic unitary groups S. Stevens applied a result of his paper with P. Broussous [BS09]. He used a map between the enlarged buildings of a centraliser and the group to apply an induction. The important property of this map is the compatibility with the Lie algebra filtrations (**CLF**) which correspond to the Moy-Prasad filtrations [MP94]. In that paper the quaternion algebra case is missing and the authors proposed a uniqueness and generalization conjecture to the reader. Functoriality questions for affine maps between Bruhat-Tits buildings have been studied before. One work to mention is Landvogt's paper [Lan00] and the paper of P. Broussous with B.Lemaire [BL02] and with S.Stevens [BS09] which I am going to explain in more detail below. For example E. Landvogt already proved in [Lan00, 2.1.1.] that for an inclusion $H \subseteq G$ of connected reductive K-groups there is toral and $H(L)$- and $\mathrm{Gal}(L|K)$-equivariant map from the enlarged Bruhat-Tits building of $H(L)$ into that of $G(L)$ such that after a normalisation of the metric of the latter building the map is isometrical. In his work he assumed $L|K$ to be a quasi-local extension.

We consider a p-adic field k_0 of residue characteristic not two, a k_0-form $\mathbf{G} := \mathbf{U}(h)$ of $\mathbf{GL}_n$, $\mathbf{Sp}_n$, or $\mathbf{O}_n$ where h is a signed hermitian form, a Lie algebra element $\beta \in \mathrm{Lie}(\mathbf{G})(k_0)$ such that $k_0[\beta]$ is semisimple and its centraliser $\mathbf{H} := \mathbf{U}(h)_\beta$. P. Broussous and S.Stevens give a model in terms of lattice functions for the enlarged Bruhat-Tits building $\mathfrak{B}^1(\mathbf{H}, k_0)$ if β is separable. This model leads them to the definition of $\mathfrak{B}^1(\mathbf{H}, k_0)$ if β is not separable. We embed $\mathfrak{B}^1(\mathbf{H}, k_0)$ into $\mathfrak{B}^1(\mathbf{G}, k_0)$ by an affine, $\mathbf{H}(k_0)$-equivariant CLF-map j. In [BL02] such a map was fully studied in the other case of $\mathbf{GL}_n(D)$ instead of $\mathbf{U}(h)$ and in [BS09] the authors considered the case where the image of h is a field and k_0 has an odd residual characteristic. In the latter paper the authors showed that if β is non-zero and generates a field then the CLF-property determines j. In this thesis we consider the general case, more precisely we include the quaternion algebra case and we analyse uniqueness without

any further restriction on β. The group $\mathbf{H}$ decomposes under β into classical groups $\mathbf{H}_i$. We construct the map j such that it has the above properties, see theorem 3.26, and we prove at first in theorem 4.9 that there is no other CLF-map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}^1(\mathbf{G}, k_0)$ if the groups $\mathbf{H}_i$ are unitary, i.e. of the form $\mathbf{U}(h_i)$, and not k_0-isomorphic to the isotropic $\mathbf{O}_2$. Secondly we show that in general a $\mathrm{Z}(\mathbf{H}^0(k_0))$-equivariant, affine CLF-map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}^1(\mathbf{G}, k_0)$ has to be unique up to a translation of the building $\mathfrak{B}^1(\mathbf{H}, k_0)$, see 4.24. A summary of the theorems of part one is given in chapter 6. For the buildings we use the model with lattice functions which are introduced in [BL02] and [BS09].

The aim of chapter 1 is to give the exact definition of $\mathbf{GL}_D(V)$ and $\mathbf{U}(h)$. We also repeat the notion of a signed hermitian form and a Witt decomposition.

Chapter 2 relies heavily on results which are summarised in the appendix. The second aim of this work is to give a complete definition of the Bruhat-Tits building for $\mathbf{GL}_D(V)$ over a p-adic field and for $\mathbf{U}(h)$ over a p-adic field of residue characteristic not two. The way of construction is taken from the articles of Bruhat and Tits. We give the definition of several kinds of lattice functions and shortly introduce the Lie algebra filtrations.

For the next two chapters we fix a separable Lie algebra element until section 4.5. In chapter 3 the section 3.1 is devoted to the definition of the CLF-property. After recalling results of [BL02] we prove the existence of a CLF-map in the case of $\mathbf{U}(h)$. The proof of the torality of the constructed map is given in chapter 5.

Chapter 4 provides the proof of the uniqueness results stated above and in section 4.6 we show how the preceding results of chapter 3 and 4 generalise to the case of a non-separable Lie algebra element.

Part 2

In the whole part 2 we consider a finite dimensional skewfield D with centre a p-adic field F. Embedding types were introduced in the paper of Broussous and Grabitz [BG00]. They considered one step on the way to construct the smooth dual of $G := \mathrm{GL}_m(D)$ using Bushnell and Kutzko's strategy [BK93] for $\mathrm{GL}_n(F)$. The aim is to produce a list of possible candidates for simple types, i.e. a list of pairs (J, λ) consisting of a compact mod center subgroup J and a smooth irreducible representation of J with two properties. The second property states that if two paires are contained in the same irreducible representation of G then they are conjugate under the action of G. The idea is to construct the list of (J, λ) by an inductive procedure using simple strata. A simple stratum is a quadruple $[\mathfrak{a}, n, q, \beta]$, especially consisting of a hereditary order $\mathfrak{a}$ normalised by an element β of $A := \mathrm{M}_m(D)$ which generates a field E. To prove the second property above they needed a rigidity for simple strata relying on a description of the way $E|F$ is embedded in A. This was done by introducing numerical invariants.

Let $E_D|F$ be the maximal unramified subextension of $E|F$ which can be embedded in D. In part 2 we show how to obtain the embedding type of $(E, \mathfrak{a})$ if one applies

$$j_{E_D} : \mathfrak{B}(\mathbf{GL}_m(D), F)^{E_D^\times} \to \mathfrak{B}(Z_{\mathbf{GL}_m(D)}(E_D), E_D)$$

on the barycenter $M_\mathfrak{a}$ of $\mathfrak{a}$ (see 11.3). The inverse of j_{E_D} is the unique CLF-map from the latter building into $\mathfrak{B}(\mathbf{GL}_m(D), F)$. The map was constructed and analysed by Broussous and Lemaire in [BL02].

In chapter 8 we recall the definition of embedding type and we introduce the easy numerical tools which enables us to decode the embedding type from $y := j_{E_D}(M_{\mathfrak{a}})$.

The aim of chapter 9 is to describe the simplicial structure of $\mathfrak{B}(\mathbf{GL}_m(D), F)$ in terms of lattice chains as it has been done in [BL02].

In chapter 10 we state the connection between the oriented barycentric coordinates of y, i.e. the so called local type of y, and the embedding type of $(E, \mathfrak{a})$.

General notation

1. All rings we consider in this thesis are unital.

2. The set of natural numbers starts with 1 and the set of the first r natural numbers is denoted by $\mathbb{N}_r$. For the set of non-negative integers we use the symbol $\mathbb{N}_0$ and the set of its first $r + 1$ elements is writen as $\mathbb{N}_0^r$.

3. If k is a non-Archimedean local field with valuation ν we denote by
   - o_k the valuation ring of k,
   - $\mathfrak{p}_k$ the valuation ideal of k and by
   - κ_k the residue field of k.

4. For an arbitrary field k we fix an algebraic closure $\bar{k}$ and the maximal inseparable (resp. separable) field extension of k in $\bar{k}$ is denoted by k^{isep} (resp. k^{sep}).

5. If we have fixed a local field (k, ν) we also write ν for the unique extension of ν to D for any finite dimensional skewfield extension $D|k$. We also use the notation $o_D, \mathfrak{p}_D$ and κ_D. We write $e(D|k)$ for the ramification index and $f(D|k)$ for the inertia degree.

6. The symbol $\mathrm{Z}(N)$ denotes the center of N and we write $\mathrm{Z}_N(M)$ for the centraliser of M in N for a set N with multiplication and a subset M of N.

7. The $m \times m$ identity matrix is denoted by $\mathbb{1}_m$.

Part I.

Canonical maps, buildings and centralisers

We fix a field k. We have the following conventions on k.

- In section 1.1.2 and 1.2 the characteristic of k is not two.

- In this part from chapter 2 on we assume k to be a non-Archimedian local field with discrete valuation ν.

- In this part from section 2.3 on we further assume k to have residue characteristic not two.

1. Classical groups

1.1. Algebraic preliminaries

1.1.1. Semisimple algebras

In this subsection we do not need any restriction on the characteristic of k. A finite dimensional k-algebra A is *simple* if there is no ideal of A which is different from $\{0\}$ and A, and it is semisimple if A has no nilpotent ideal except the zero-ideal.

Theorem 1.1 *(Wedderburn) If A is a finite dimensional semisimple k-algebra there is a unique natural number m and an m-tuple*

$$(p_1, \ldots, p_m)$$

of pairs

$$p_i = ([D_i], n_i)$$

consisting of a k-algebra isomorphism class of a skewfield D_i and a natural number n_i such that there is a k-algebra isomorphism

$$A \cong \prod_{i=1}^{m} \mathrm{M}_{n_i}(D_i).$$

Up to permutation the m-tuple $(p_1, \ldots, p_m)$ is uniquely determined by A.

Remark 1.2 1. If A in the theorem is commutative it is k-isomorphic to a product of fields.

2. A finite dimensional simple k-algebra is k-isomorphic to a matrix ring, because A is unital by our general notation.

Definition 1.3 A finite dimensional k-algebra is *separable* if for every field extension $L|k$ the L-algebra $A \otimes_k L$ is semisimple.

Remark 1.4 A commutative finite dimensional k-algebra is separable if and only if it is k-isomorphic to a product of separable extension fields of k.

Definition 1.5 An element β of a finite dimensional k-algebra A is *separable* if the k-algebra $k[\beta]$ is separable.

Definition 1.6 A triple (A, V, D) consisting of:

- a skewfield D which is a finite dimensional k-algebra

- a finite dimensional right D-vector space V

- and $A := \mathrm{End}_D(V)$

is called a *simple k-datum.* Such a datum is *central* if the center of D is k. If we want to emphasize $m := \dim_D V$ and $d := \deg(D)$ then we write (A, V, m, D, d) instead of (A, V, D) and if in addition we need a maximal Galois extension L of k in D we write $(A, V, m, D, d, L|k)$. A simple datum is *local* if k is a non-Archimedean local field.

1.1.2. Algebras with involution

For references we recommend the books [Knu98], [KMRT98] and [Sch85]. We assume $\mathrm{char}(k) \neq 2$.

An *involution* of a ring is an antimultiplicative ring automorphism of order 1 or 2.

Notation 1.7   1. If σ is an involution on the ring R, we introduce the following standard notation.

$$\mathrm{Sym}(R, \sigma) := \{r \in R \mid \sigma(r) = r\}$$

$$\mathrm{Skew}(R, \sigma) := \{r \in R \mid \sigma(r) = -r\}$$

for the set of symmetric and the set of skewsymmetric elements of R.

2. For a semisimple finite dimensional k-algebra A with involution σ we denote by

$$\mathrm{U}(\sigma) := \{a \in A^\times \mid \sigma(a)a = 1\}$$

the *unitary group* of (A, σ) and by

$$\mathrm{SU}(\sigma) := \{a \in \mathrm{U}(\sigma) \mid \mathrm{Nrd}(a) = 1\}$$

the *special unitary* group of (A, σ).

The symbol Nrd denotes the reduced norm of A over k.

Definition 1.8 Let A be a simple central finite dimensional k-algebra. An involution σ of A is *of the first kind* if it fixes every element of $\mathrm{Z}(A)$ and *of the second kind* otherwise. We call an involution of the first kind on A *orthogonal* if

$$\dim_k \mathrm{Sym}(A, \sigma) > \dim_k \mathrm{Skew}(A, \sigma)$$

and *symplectic* otherwise. An involution of the second kind is also called *unitary*.

Remark 1.9 For a symplectic involution on A we have

$$\dim_k \mathrm{Sym}(A, \sigma) < \dim_k \mathrm{Skew}(A, \sigma)$$

and for a unitary involution we get an equality.

Assumption 1.10 *For this section we fix a central simple k-datum (A, V, m, D, d). We assume that there is an involution ρ on D.*

One way to obtain an involution on A is to take the adjoint involution of an ϵ-hermitian form.

Definition 1.11 Fix an ϵ which is 1 or -1. An *ϵ-hermitian form on V* is a biadditive map h from $V \times V$ to D such that

1. $h(v, w) = \epsilon\rho(h(w, v))$ for all $v, w \in V$ and

2. h is sesquilinear in the first coordinate and linear in the second coordinate, i.e.

$$h(wd_1, vd_2) = \rho(d_1)h(w, v)d_2,$$

   and

3. it is non-degenerate.

The pair (V, h) is called an *ϵ-hermitian space.* An ϵ-hermitian form h_1 and a δ-hermitian form h_2 are equivalent if there is an element $b \in k^\times$ such that $bh_1 = h_2$. An involution σ of $\mathrm{End}_D(V)$ is called *the adjoint involution* of h, and it is denoted by σ_h, if for every $a \in \mathrm{End}_D(V)$ and for every $v, w \in V$ we have

$$h(a(v), w) = h(v, \sigma(a)(w)).$$

If we do not want to state the ϵ of an ϵ-hermitian form we write *signed* hermitian form. There is a general notion of quadratic forms given in [Sch85, 7.3.3] which includes the notion of signed hermitian forms for the case of characteristic different from 2.

Proposition 1.12 *There is a bijection from the set of equivalence classes of signed hermitian forms to the set of involutions of $\mathrm{End}_D(V)$ whose restriction to $\mathrm{Z}(D)$ is ρ.*

For a given signed hermitian form h we have the map

$$\hat{h}: \ V \to V^*, \ \hat{h}(v)(w) := h(v, w),$$

where V^* is the dual vector space of V. It is an isomorphism of D-left vector spaces where D acts on V on the left via $dv := v\rho(d)$.

Proof: Equivalent signed hermitian forms have the same adjoint involution. To prove the injectivity assume $\sigma_{h_1} = \sigma_{h_2}$ and we call this involution σ. It implies that $\psi := \hat{h}_2^{-1} \circ \hat{h}_1$ is in the center of $\mathrm{End}_D(V)$, because for $a \in \mathrm{End}_D(V)$ and $v, w \in V$ we

have

$$h_2(\psi(a(v)), w) = h_1(a(v), w)$$
$$= h_1(v, \sigma(a)(w))$$
$$= h_2(\psi(v), \sigma(a)(w))$$
$$= h_2(a(\psi(v)), w).$$

Thus h_1 and h_2 are equivalent. If one chooses a D-basis (v_i) of V the map

$$h_{(v_i)} : \; (\sum_i v_i \lambda_i, \sum_i v_i \mu_i) \mapsto \sum_i \rho(\lambda_i)\mu_i$$

is a hermitian form whose adjoint involution $\sigma_{(e_i)}$ equals to ρ on $Z(D)$. We identify A with $M_m(D)$ and we have $\sigma_{(e_i)}(C) = \rho(C)^T$ where $\rho(C)$ is meant to be the matrix obtained after applying ρ to every entry of C. The surjectivity is now given by the Skolem-Noether theorem, more precisely: we take an involution σ whose restriction to $Z(D)$ is ρ. By the Skolem-Noether theorem there is a $B \in \mathrm{GL}_m(D)$ which satisfies

$$\sigma(C) = B\rho(C)^T B^{-1}$$

for all matrices C. By $\sigma^2 = \mathrm{id}$ we get that there is a $\lambda \in Z(D)$ such that $B\rho(B^{-1})^T = \lambda \mathbb{1}_m$ and $\rho(\lambda)\lambda$ is 1 by $\sigma_{(e_i)}^2 = \mathrm{id}$.

Case 1: If ρ fixes λ, the matrix B^{-1} is a Gram matrix of a λ-hermitian form with adjoint involution σ.

Case 2: If λ is not a fixed point of ρ, Hilbert's 90th theorem [Ker90, 12.3] implies the existence of an element $\alpha \in Z(D)^\times$ such that

$$\alpha\rho(\alpha)^{-1} = \lambda.$$

Thus αB^{-1} is the Gram matrix of a 1-hermitian form whose adjoint involution is σ. q.e.d.

To study groups $\mathrm{U}(\sigma)$ we need Witt's theorem.

Definition 1.13 An $r \times r$-matrix M is *antidiagonal* if all entries m_{ij} with $i + j \neq r + 1$ are zero. We denote an antidiagonal matrix M by

$$\mathrm{antidiag}(m_{r,1}, m_{r-1,2}, \ldots, m_{1,r}).$$

The following theorem uses $\mathrm{char}(k) \neq 2$.

Theorem 1.14 *(Witt) [Sch85, 7.9.2 (iii)] Let h be an ϵ-hermitian form of V. Then there is a basis (v_i) of V such that the Gram matrix $\mathrm{Gram}_{(v_i)}(h)$ of h over (v_i) has the*

form

$$
\begin{pmatrix}
0 & M & 0 \\
\epsilon M & 0 & 0 \\
0 & 0 & B
\end{pmatrix}
\tag{1.1}
$$

such that $M = \operatorname{antidiag}(1, \ldots, 1)$ and that the ϵ-hermitian form corresponding to B is anisotropic.

Definition 1.15 Under the assumptions of the theorem the size r of M does not depend on the basis (v_i). We call r the *Witt index of h*.

Definition 1.16 Let r be the Witt index of a signed hermitian form h. A set of 1-dimensional vector subspaces

$$\{V_1, V_{-1}, V_2, V_{-2}, \ldots V_r, V_{-r}\}$$

together with an anistropic D-subvector space V_0 of V such that

$$\bigoplus_i V_i = V$$

is called a *Witt decomposition* of h if for all

$$i, j \in \{1, -1, 2, -2, \ldots, r, -r\}$$

we have

1. $h(V_i, V_j) = 0$ if $i \neq -j$ and

2. $h(V_i, V_0) = 0$.

Definition 1.17 If ρ is the identity of D, implying D equals k, then a 1-hermitian form is called *symmetric bilinear form* and a -1-hermitian form is said to be a *skew symmetric* bilinear form.

Theorem 1.18 *[Sch85, 7.6.3] If we exclude the skew symmetric case from the assumptions of the last theorem, i.e. the case where $\epsilon = -1$ and $\rho = \operatorname{id}_D$, the vector space V has an orthogonal basis.*

Definition 1.19 A D-basis of V such that the Gram matrix of an ϵ-hermitian form h is of the form in theorem 1.14 such that the matrix B is diagonal is called a *Witt basis* of h.

Corollary 1.20 Under the assumptions of theorem 1.14 for every Witt decomposition of h there is a Witt basis of h such that the isotropic vectors of the basis span the isotropic lines and the other vectors together span the anisotropic vector space.

Notation 1.21 We set

$$U(h) := U(\sigma_h) \text{ and } SU(h) := SU(\sigma_h)$$

if h is an ϵ-hermitian form on V.

Definition 1.22 A *hermitian k-datum* is a tuple

$$((A, V, D), \rho, k_0, h, \epsilon, \sigma)$$

- (A, V, D) is a central simple k-datum,

- ρ is an involution of D whose set of central fixed points is k_0,

- h is an ϵ-hermitian form on V with adjoint involution σ.

A hermitian datum is *local* if k is a non-Archimedean local field and ρ is continuous under the valuation of k. In this case k_0 is local too. We write (k, ν)-datum instead of k-datum to emphasize the valuation ν on k.

1.2. Forms of classical groups

A good introduction in the theory of classical groups can be found in [PR94]. We only consider $\mathrm{char}(k) \neq 2$. Let us fix a natural number n.

Notation 1.23 $\mathbf{A}^n$ denotes the affine space of dimension n. The groups $\mathbf{GL}_n$ (resp. $\mathbf{SL}_n$) are the general linear group (resp. the special linear group). All are considered as affine algebraic group schemes defined over the prime field of k.

Definition 1.24 We consider the transposition $()^T$ on $\mathbf{GL}_n(\bar{k})$. We denote by $\mathbf{O}_n$ the *orthogonal group*, i.e the subscheme of $\mathbf{GL}_n$ which is defined by the equation $g^T g = 1$, and by $\mathbf{Sp}_{2n}$ the *symplectic group*, i.e. the subscheme of $\mathbf{GL}_{2n}$ given by the equation

$$g^T J g = J,$$

where J is

$$\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix} \tag{1.2}$$

and $M := \mathrm{antidiag}(1, \dots, 1)$. The *special orthogonal group* $\mathbf{SO}_n$ is the intersection of $\mathbf{O}_n$ with $\mathbf{SL}_n$.

In this section we use the notion of a k-form and therefore we give a general definition here.

Definition 1.25 An algebraic group defined over k is called a *k-group*. Two k-groups are *k-isomorphic to each other* if there is an isomorphism of algebraic groups defined over k between them. Let $L|k$ be a field extension in $\bar{k}|k$ and let G be an L-group. A k-group H is a k-form of G if H and G are $\bar{k}$-isomorphic. A k-form H of G is an $L|k$-form of G if G and H are L-isomorphic to each other.

Convention 1.26 Instead of "isomorphic as algebraic groups" we only write "isomorphic".

Definition 1.27 A *classical group in the strict sense* is an algebraic group which is $\bar{k}$-isomorphic to $\mathbf{SL}_n(\bar{k})$, $\mathbf{SO}_n(\bar{k})$ or $\mathbf{Sp}_{2n}(\bar{k})$.

Proposition 1.28 *Let V be an n-dimensional $\bar{k}$-vector space equipped with a non-degenerate symmetric or alternate bilinear form h. We assume that V has a k-structure V_k, i.e. $V_k \otimes_k \bar{k} = V$, and that*
$$\sigma_h = \sigma \otimes_k \bar{k}$$
for an involution σ of $\mathrm{End}_k(V_k)$ of the first kind. Then the following holds.

1. *If h is alternate then the group $\mathrm{U}(h)$ equals $\mathrm{SU}(h)$ and is a $k|k$-form of $\mathbf{Sp}_n(\bar{k})$.*

2. *If h is symmetric then the group $\mathrm{SU}(h)$ (resp. $\mathrm{U}(h)$) is a $k^{sep}|k$-form of $\mathbf{SO}_n(\bar{k})$ (resp. $\mathbf{O}_n(\bar{k})$).*

Remark 1.29 The set $\mathrm{End}_{\bar{k}}(V)$ is made to an affine space defined over k in taking a k-basis of $\mathrm{End}_k(V_k)$ and introducing coordinates, i.e. we have

$$\mathrm{End}_{\bar{k}}(V) \cong \mathbf{A}^{n^2}(\bar{k})$$

Every k-linear isomorphism of $\mathrm{End}_k(V_k)$ induces a $\bar{k}$-linear isomorphism of $\mathbf{A}^{n^2}(\bar{k})$ defined over k. The composition of maps in $\mathrm{End}_{\bar{k}}(V)$ coinsides with a k-morphism

$$\phi : \ \mathbf{A}^{n^2}(\bar{k}) \times \mathbf{A}^{n^2}(\bar{k}) \to \mathbf{A}^{n^2}(\bar{k})$$

and $(\mathbf{A}^{n^2}(\bar{k}), \phi)$ is k-isomorphic to $(\mathrm{M}_n(\bar{k}), \circ)$. We identify the groups $\mathrm{U}(h)$ and $\mathrm{SU}(h)$ of the proposition with the corresponding subsets of $\mathbf{A}^{n^2}(\bar{k})$. The assertions of the proposition are valid for any choice of the basis of $\mathrm{End}_k(V_k)$.

Proof: (of proposition 1.28) By proposition 1.12 there is a $\lambda \in \bar{k}$ such that λh maps $V_k \times V_k$ to k. Without loss of generality we assume that λ is one.

1. By theorem 1.14 there is a k-basis of V_k such that the Gram matrix G of h is J from definition 1.24. Thus all elements of $\mathrm{U}(h)$ have determinant 1 and there is a k-isomorphism from $\mathbf{A}^{n^2}(\bar{k})$ to $\mathrm{M}_n(\bar{k})$ which maps $\mathrm{U}(h)$ onto $\mathbf{Sp}_n(\bar{k})$. Thus $\mathrm{U}(h)$ and $\mathrm{SU}(h)$ equal and are $k|k$-forms of $\mathbf{Sp}_n(\bar{k})$.

2. By theorem 1.18 there is an orthogonal basis of V_k with respect to $h|_{V_k \times V_k}$, i.e. the corresponding Gram matrix G of h is diagonal and has entries in k. It implies that there is a k-isomorphism from $\mathbf{A}^{n^2}(\bar{k})$ to $\mathrm{M}_n(\bar{k})$ which maps $\mathrm{U}(h)$ (resp. $\mathrm{SU}(h)$) to $\mathrm{U}(\tilde{h})$ (resp. $\mathrm{SU}(\tilde{h})$) where

$$\tilde{h} : \bar{k}^n \times \bar{k}^n \to \bar{k}$$

is a bilinear form whose Gram matrix under the standard basis of $\bar{k}^n$ is G. The characteristic of k is not 2 and thus the roots of the diagonal elements of G are separable over k, i.e. there is a diagonal matrix X in $\mathrm{M}_n(k^{sep})$ such that $XX = G$. The inner $\bar{k}$-algebra automorphism $\mathrm{Inn}(X)$ maps $\mathrm{U}(\tilde{h})$ (resp. $\mathrm{SU}(\tilde{h})$) onto $\mathbf{O}_n(\bar{k})$ (resp. $\mathbf{SO}_n(\bar{k})$). This map is a k^{sep}-isomorphism. $\mathbf{O}_n$ and $\mathbf{SO}_n$ are defined over the prime field and therefore defined over k^{sep}. Thus $\mathrm{U}(h)$ and $\mathrm{SU}(h)$ are defined over k^{sep}. Both groups are k-closed too, and since $k^{sep}|k$ is a separable field extension they are defined over k.

q.e.d.

Assumption 1.30 *We fix a central simple k-datum $(A, V, m, D, d, L|k)$ and we assume L to be a subfield $\bar{k}$.*

We now give forms of $\mathbf{SL}_n(\bar{k})$, $\mathbf{SO}_n(\bar{k})$ and $\mathbf{Sp}_n(\bar{k})$. We also write $\mathrm{GL}_D(V)$ for $\mathrm{Aut}_D(V)$ and $\mathrm{SL}_D(V)$ for the set of elements of $\mathrm{End}_D(V)$ with reduced norm one.

Proposition 1.31 *The group $GL_D(V)$ (resp. $SL_D(V)$) is the set of k-rational points of an $L|k$-form of $\mathbf{GL}_{md}(\bar{k})$ (resp. $\mathbf{SL}_{md}(\bar{k})$) denoted by $\mathbf{GL}_D(V)$(resp. $\mathbf{SL}_D(V)$).*

Proof: We fix a k-basis of $\mathrm{End}_D(V)$ to introduce coordinates. We identify the tensor product $\mathrm{End}_D(V) \otimes_k \bar{k}$ with $\mathbf{A}^{m^2 d^2}(\bar{k})$, i.e. the set of k-rational points of $\mathbf{A}^{m^2 d^2}$ is identified with $\mathrm{End}_D(V)$. $\mathrm{End}_D(V) \otimes_k \bar{k}$ is L-isomorphic to $\mathrm{M}_{md}(\bar{k})$ as a $\bar{k}$-algebra, because L is a splitting field of D. The reduced norm corresponds to the determinant on $\mathrm{M}_{md}(\bar{k})$ and thus there is an L-isomorphism from $\mathbf{A}^{m^2 d^2 + 1}(\bar{k})$ to $\mathrm{M}_{md}(\bar{k}) \times \bar{k}$ which maps

$$\mathbf{GL}_D(V) := \{(g, y) \in (\mathrm{End}_D(V) \otimes_k \bar{k}) \times \bar{k}|\ \mathrm{Nrd}(g)y = 1\}$$

onto $\mathbf{GL}_{md}(\bar{k})$ and

$$\mathbf{SL}_D(V) := \{g \in \mathbf{GL}_D V|\ \mathrm{Nrd}(g) = 1\}$$

onto $\mathbf{SL}_{md}(\bar{k})$.

The reduced norm is a homogeneous polynomial in $k[X_1, \ldots, X_{(md)^2}]$ and therefore $\mathbf{GL}_D(V)$ and $\mathbf{SL}_D(V)$ are k-closed. $\mathbf{GL}_{md}$ and $\mathbf{SL}_{md}$ are defined over the prime field and therefore defined over L. Thus $\mathbf{GL}_D(V)$ and $\mathbf{SL}_D(V)$ are defined over L. The separability of $L|k$ implies that both groups are defined over k. q.e.d.

Remark 1.32 If F is an intermediate field between k and $\bar{k}$ then the set of F-rational points of $\mathbf{GL}_D(V)$ is given by $(\mathrm{End}_D(V) \otimes_k F)^{\times}$.

Assumption 1.33 *We now extend our assumption and fix a hermitian k-datum*

$$((A, V, m, D, d, L|k), \rho, k_0, h, \epsilon, \sigma).$$

Proposition 1.34 *[PR94, 2.15] There is an algebraic group* $\mathbf{U}(h)$ *(resp.* $\mathbf{SU}(h)$*) which is*

1. *an* $L|k$*-form of* $\mathbf{Sp}_{md}(\bar{k})$ *if* σ *is symplectic,*

2. *a* $k^{sep}|k$*-form of* $\mathbf{O}_{md}(\bar{k})$ *(resp.* $\mathbf{SO}_{md}(\bar{k})$*) if* σ *is orthogonal and*

3. *an* $L|k_0$*-form of* $\mathbf{GL}_{md}(\bar{k})$ *(resp.* $\mathbf{SL}_{md}(\bar{k})$*) if* σ *is unitary*

such that its set of k-rational points is $\mathrm{U}(h)$ *(resp.* $\mathrm{SU}(h)$*).*

Definition 1.35 We also denote $\mathbf{U}(h)$ and $\mathbf{SU}(h)$ by $\mathbf{U}(\sigma_h)$ and $\mathbf{SU}(\sigma_h)$ respectively.

For the orthogonal and the unitary case in the proposition we needed that the characteristic of k is different from two. In the unitary case $k|k_0$ has degree two.

   Proof: At first we assume that σ is of the first kind and we define

$$\mathbf{U}(h) := \{g \in \mathbf{GL}_D V| \ g^{(\sigma \otimes_k \mathrm{id})}g = 1\}$$

and

$$\mathbf{SU}(h) := \{g \in \mathbf{U}(h)| \ \mathrm{Nrd}(g) = 1\}.$$

The assertions (1) and (2) follow now from proposition 1.28, because $\mathrm{End}_D(V) \otimes_k \bar{k}$ is L-isomorphic to $\mathrm{M}_{md}(\bar{k})$.

   At second we assume that σ is of the second kind. Here we use the notion of the Weil-restriction, see appendix chapter A. We have

$$\mathrm{Res}_{k|k_0}(\mathrm{End}_D(V) \otimes_k \bar{k}) = \mathrm{End}_D(V) \otimes_{k_0} \bar{k}$$

and a commutative diagram

$$
\begin{array}{ccc}
\mathrm{End}_D(V) \otimes_{k_0} \bar{k} & \cong & (\mathrm{End}_D(V) \otimes_k \bar{k}) \times (\mathrm{End}_D(V) \otimes_k \bar{k})^\rho \\
\cup & & \cup \\
\mathrm{Res}_{k|k_0}(\mathbf{A}^1)(\bar{k}) & \cong & \mathbf{A}^1(\bar{k}) \times \mathbf{A}^1(\bar{k})
\end{array}
$$

where the k-isomorphism on the top is induced by

$$\lambda \otimes_{k_0} \mu \in k \otimes_{k_0} \bar{k} \mapsto (\lambda\mu, \rho(\lambda)\mu).$$

We now explain $(\mathrm{End}_D(V) \otimes_k \bar{k})^\rho$. Under the choice of a k-basis in $\mathrm{End}_D(V)$ the composition of endomorphisms is given by a morphism

$$\phi: \ \mathbf{A}^{(md)^2} \times \mathbf{A}^{(md)^2} \to \mathbf{A}^{(md)^2}$$

defined over k. The multiplication on $(\mathrm{End}_D(V) \otimes_k \bar{k})^\rho$ is given by ϕ^ρ. The involution $\sigma \otimes_{k_0} \mathrm{id}$ defines an involution $\tilde{\sigma}$ on the right side of the diagram which permutes the coordinates of $\bar{k} \times \bar{k}$. Thus there is an L-isomorphism from $(\mathrm{End}_D(V) \otimes_k \bar{k}) \times (\mathrm{End}_D(V) \otimes_k \bar{k})^\rho$ to $\mathrm{M}_{md}(\bar{k}) \times \mathrm{M}_{md}(\bar{k})$ such that the involution $\tilde{\sigma}$ corresponds to the involution

$$\sigma' : (B_1, B_2) \mapsto (B_2^T, B_1^T).$$

The group $\mathrm{U}(\sigma')$ is a $k|k$-form of $\mathbf{GL}_{md}$ and thus

$$\mathbf{U}(h) := \{g \in \mathrm{Res}_{k|k_0}(\mathbf{GL}_D V) |\ g^{\mathrm{Res}_{k|k_0}(\sigma \otimes_k \mathrm{id})} g = 1\}$$

is an $L|k_0$-form of $\mathbf{GL}_D(V)$. The Weil-restriction of the reduced norm from k to k_0 corresponds to $\det \times \det$ on $\mathrm{M}_{md}(\bar{k}) \times \mathrm{M}_{md}(\bar{k})$ and we conclude that under

$$\mathbf{U}(h) \cong \mathbf{GL}_{md}$$

the group

$$\mathbf{SU}(h) := \{g \in \mathbf{U}(h) |\ \mathrm{Res}_{k|k_0}(\mathrm{Nrd})(g) = 1\}$$

is mapped onto $\mathbf{SL}_{md}(\bar{k})$. The groups fulfill

$$\mathbf{U}(h)(k_0) = \mathrm{U}(h) \text{ and } \mathbf{SU}(h)(k_0) = \mathrm{SU}(h)$$

by definition. q.e.d.

Remark 1.36 For a version of a converse of this theorem see [KMRT98, 26.9, 26.12, 26.14, 26.15].

Later we only consider non-Archimedean local fields and there we have the following restricted possibility for the involution.

Theorem 1.37 *[Sch85, 10.2.2] If k is a non-Archimedean local field and ρ is an involution of D then if ρ is of the second kind we have $D = k$ and if ρ is of the first kind, the degree of D is not bigger than two.*

2. Bruhat-Tits building of a Classical group

For an introduction to the theory of buildings, I recommand [AB08] or the previous version [Bro89].

Assumption 2.1 *In this chapter we fix a local simple k-datum*

$$(A, V, m, D, d, L|k),$$

especially we assume that k is a non-Archimedean local field with valuation ν. The unique extension of ν to a finite dimesional skewfield over k is also denoted by ν. We denote $\mathbf{GL}_D(V)$ by $\tilde{\mathbf{G}}$ and its set of k-rational points by $\tilde{G}$, i.e. $\tilde{G} := GL_D(V)$.

2.1. Norms and lattice functions

This is a collection of definitions and results of [BL02] and [BT84b].

2.1.1. First definitions

Definition 2.2 A *D-norm* on V is a map $\alpha :\ V \to \mathbb{R} \cup \{\infty\}$ such that for all $t \in D$ and $v, v' \in V$ we have

1. $\alpha(v) = \infty \implies v = 0$,

2. $\alpha(tv) = \alpha(v) + \nu(t)$ and

3. $\alpha(v + v') \geq \min(\alpha(v), \alpha(v'))$.

The set of D-norms on V is denoted by $\mathrm{Norm}_D^1(V)$.

Given a norm the family of balls around 0 is a decreasing function of lattices. We recall the definition below.

Definition 2.3 A finitely generated o_D-submodule Γ of V is called a *(full) o_D-lattice of V* if $\mathrm{span}_D(\Gamma) = V$. We denote by $\mathrm{Latt}(V, o_D)$ the set of all full o_D-lattices of V.

Definition 2.4 [BL02, 2.1] A map $\Lambda :\ \mathbb{R} \to \mathrm{Latt}(V, o_D)$ is called an *o_D-lattice function,* if for all reals r and s with $r \leq s$ we have

1. $\Lambda(r + \nu(\pi_D))$ equals $\Lambda(r)\pi_D$,

2. $\Lambda(r)$ contains $\Lambda(s)$ and

3. the lattice $\Lambda(r)$ is the intersection of the $\Lambda(r - \epsilon)$ where ϵ runs over all positive real numbers, i.e. Λ is left continuous, when $\mathrm{Latt}(V, o_D)$ is endowed with the discrete topology.

The set of o_D-lattice functions is denoted by $\mathrm{Latt}^1_{o_D}(V)$. For $\Lambda \in \mathrm{Latt}^1_{o_D}(V)$ the number of elements in $\Lambda([0, \nu(\pi_D)[)$ is the *(simplicial) rank* of Λ.

Remark 2.5 [BL02, I.2.4] The map $\alpha \mapsto \Lambda_\alpha$ with

$$\Lambda_\alpha(r) := \{x \in V|\ \alpha(x) \geq r\}$$

is a $\tilde{G}$-set isomorphism from $\mathrm{Norm}^1_D(V)$ to $\mathrm{Latt}^1_{o_D}(V)$ with actions

$$g.\alpha := \alpha \circ g^{-1} \text{ and } (g.\Lambda)(r) := g(\Lambda(r)).$$

In certain proofs it is useful to reduce the m-dimensional case to lower dimensional cases. This is done with the concept of a splitting vector space decomposition.

Definition 2.6 [BT84b, 1.4] A family $(V^1, \ldots, V^l)$ of D-vector subspaces is a *splitting decomposition of V* for $\alpha \in \mathrm{Norm}^1_D(V)$, if

- $V = \oplus_i V^i$ and

- for all $(v_i) \in \prod V^i$ we have

$$\alpha(\sum_i v_i) = \min_i \alpha(v_i).$$

A family (V^i) is a *splitting decomposition of V* for an o_D-lattice function Λ on V if

$$\Lambda(t) = \oplus_i (\Lambda(t) \cap V^i)$$

holds for all t. We also say that the norm or the lattice function is split by $(V^i)_i$. If all V^i are one dimensional and $(b_i) \in \prod_i V^i$ is a D-basis of V we call it a *splitting basis* for norms and lattice functions which are split by (V^i).

Definition 2.7 [BT84b, 1.11 (17)] The *dual of α* is the o_D-norm α^* on

$$V^* := \mathrm{Hom}_D(V, D)$$

defined by

$$\alpha^*(f) := \inf\{\nu(f(v)) - \alpha(v)|\ v \in V \setminus \{0\}\}.$$

Proposition 2.8 *[BT84b, 1.11 (18), 1.26]*

1. *The dual basis of a splitting basis $(v_i)_i$ of α is a splitting basis of α^* and the equation $\alpha^*(v_i^*) = -v_i$ holds.*

2. Any two o_D-norms on V have a common splitting basis.

The following definition generalises the definition [BT72, 1.1.1].

Definition 2.9 (Affine structure) An *affine structure* on a set S is a function

$$a : S \times S \times [0,1] \to S.$$

We write

$$ts + (1-t)s' := a(s, s', t).$$

Definition 2.10 For a real number x we denote by $[x]+$ the smallest integer which is not smaller than x.

Remark 2.11 We have an affine structure on $\mathrm{Latt}^1_{o_D}(V)$. If Λ and Λ' are two elements of $\mathrm{Latt}^1_{o_D}(V)$ with a common splitting basis (v_i), i.e. there are m-tupels (α_i) and (β_i) of real numbers such that

$$\Lambda(x) = \bigoplus_i v_i \mathfrak{p}_D^{[(x-\alpha_i)d]+}$$

and

$$\Lambda'(x) = \bigoplus_i v_i \mathfrak{p}_D^{[(x-\beta_i)d]+},$$

then for $\lambda \in [0,1]$ we define a new element of $\mathrm{Latt}^1_{o_D}(V)$ by

$$(\lambda\Lambda + (1-\lambda)\Lambda')(x) := \bigoplus_i v_i \mathfrak{p}_D^{[(x-\lambda\alpha_i-(1-\lambda)\beta_i)d]+}.$$

This definition does not depend on the choice of the basis (v_i).

Remark 2.12 Let V' be another finite dimensional right D-vector space. The map

$$\mathrm{Latt}^1_{o_D}(V) \times \mathrm{Latt}^1_{o_D}(V') \to \mathrm{Latt}^1_{o_D}(V \oplus V')$$

given by

$$(\Lambda, \Lambda') \mapsto \Lambda \oplus \Lambda'$$

with

$$(\Lambda \oplus \Lambda')(t) := \Lambda(t) \oplus \Lambda'(t)$$

is affine and $\tilde{G} \times \mathrm{GL}_D(V')$-equivariant.

Definition 2.13 The lattice function $\Lambda \oplus \Lambda'$ is called the *direct sum* of Λ and Λ'

Remark 2.14 Under $\alpha \mapsto \Lambda_\alpha$ the affine structure of $\mathrm{Latt}^1_{o_D}(V)$ defines the following affine structure on $\mathrm{Norm}^1_D(V)$. For $\lambda \in [0,1]$, α, $\alpha' \in \mathrm{Norm}^1_{o_D}(V)$ and a common splitting basis $(v_1, \ldots, v_n)$ we have that

$$(\lambda\alpha + (1-\lambda)\alpha')$$

is the norm with splitting basis $(v_i)_i$ such that

$$v_i \mapsto \lambda \alpha(v_i) + (1 - \lambda)\alpha'(v_i).$$

2.1.2. Square lattice functions

Assumption 2.15 *Let us now assume that k is the center of D.*

We also have k-norms and k-lattice functions on A. We recall that for an o_D-lattice function Λ (resp. D-norm α) on V the o_k-lattice function

$$r \mapsto \mathrm{End}(\Lambda)(r) := \{a \in A \mid a(\Lambda(s)) \subseteq \Lambda(s + r) \ \forall s \in \mathbb{R}\}$$

(resp. k-norm

$$a \mapsto \mathrm{End}(\alpha)(a) := \inf\{\alpha(a(x)) - \alpha(x) \mid x \in V \setminus \{0\}\})$$

on A is called *square lattice function* (resp. *square norm*) on A. The set of square lattice functions, square norms on A is denoted by $\mathrm{Latt}^2_{o_k}(A)$, $\mathrm{Norm}^2_k(A)$ respectively. There is a $\tilde{G}$-actions on $\mathrm{Norm}^2_k(A)$, $\mathrm{Latt}^2_{o_k}(A)$ which is given by

$$g\beta := \beta \circ \mathrm{Inn}(g^{-1}), \ (g\Gamma)(r) := \mathrm{Inn}(g)(\Gamma(r))$$

respectively where Inn denotes the adjoint action of $\tilde{G}$ on A.

Remark 2.16 [BL02, 4.10] The map $\mathrm{End}(\alpha) \mapsto \mathrm{End}(\Lambda_\alpha)$ is a $\tilde{G}$-set isomorphism from $\mathrm{Norm}^2_k(A)$ to $\mathrm{Latt}^2_{o_k}(A)$.

A square lattice functions encodes the o_D-lattice function up to translation.

Definition 2.17 The *translation of an o_D-lattice function* Λ by a real number s is defined as

$$(\Lambda + s)(t) := \Lambda(t - s).$$

Two o_D-lattice functions Λ and Λ' are *equivalent* if Λ is a translation of Λ'. The set of equivalence classes of o_D-lattice functions of V is denoted by $\mathrm{Latt}_{o_D}(V)$. Taking classes in remark 2.11 one obtains an affine structure for $\mathrm{Latt}_{o_D}(V)$, i.e.

$$\lambda[\Lambda] + (1 - \lambda)[\Lambda'] := [\lambda\Lambda + (1 - \lambda)\Lambda'], \ \lambda \in [0, 1].$$

The *translation of an o_D-norm* α of V by an element s of $\mathbb{R}$ is defined as

$$(\alpha + s)(v) := \alpha(v) + s.$$

Two norms are *equivalent* if one is a translation of the other and the set of all equivalence classes is denoted by $\mathrm{Norm}_D(V)$.

Theorem 2.18 *[BL02, I.4] The following is a commutative diagram of $\tilde{G}$-set isomorphisms.*

$$
\begin{array}{ccc}
\mathrm{Norm}_D(V) & \rightarrow & \mathrm{Latt}_{o_D}(V) \\
\downarrow & & \downarrow \\
\mathrm{Norm}_k^2(A) & \rightarrow & \mathrm{Latt}_{o_k}^2(A)
\end{array}
\tag{2.1}
$$

The maps are defined as follows.

- *on the top: $[\alpha] \mapsto [\Lambda_\alpha]$,*

- *on the bottom: $\beta \mapsto \Lambda_\beta$,*

- *on the left: $[\alpha] \mapsto \mathrm{End}(\alpha)$,*

- *on the right: $[\Lambda] \mapsto \mathrm{End}(\Lambda)$.*

For sake of completeness we give a second diagram.

Remark 2.19 The map given in remark 2.5 induces a commutative diagram of $\tilde{G}$-set isomorphisms.

$$
\begin{array}{ccc}
\mathrm{Norm}_D^1(V) & \rightarrow & \mathrm{Latt}_{o_D}^1(V) \\
\downarrow & & \downarrow \\
\mathrm{Norm}_D(V) & \rightarrow & \mathrm{Latt}_{o_D}(V)
\end{array}
\tag{2.2}
$$

The maps downwards send an element to its equivalence class.

We give a last remark describing the behavior of square lattice functions under direct sum. We use the assumptions of remark 2.12.

Definition 2.20 For $a \in A$ and $a' \in \mathrm{End}_D(V')$ the direct sum of a and a' in $\mathrm{End}_D(V \oplus V')$ is defined by

$$
(a \oplus a')(v, v') := (a(v), a'(v')).
$$

Proposition 2.21

$$
\mathrm{End}(\Lambda \oplus \Lambda')(t) \cap (\mathrm{End}_D(V) \oplus \mathrm{End}_D(V')) = \mathrm{End}(\Lambda)(t) \oplus \mathrm{End}(\Lambda')(t).
$$

2.2. The Bruhat-Tits building of $\mathrm{GL}_D(V)$ over k

We consider the building of the following valuated root datum mentioned in [BT84b]. We briefly repeat the construction. As usual $X^*(?)_k$ and $X_*(?)_k$ denote the set of k-rational characters and cocharacters respectively.

Assumption 2.22 *In this section we assume that $k = \mathrm{Z}(D)$.*

We take a D-basis (v_i) of V and consider the maximal k-split torus T of $\tilde{\mathbf{G}}$ whose set of k-rational points is

$$
\{t \in \tilde{G} \mid tv_i \in kv_i, \text{ for all } i\}.
$$

2. Bruhat-Tits building of a Classical group

With the basis $\mathrm{GL}_D(V)$ identifies with $\mathrm{GL}_m(D)$ and the k-rational points of T are diagonal matrices. The torus acts on the Lie algebra by conjugation and the k-rational roots of T are the characters

$$t \mapsto a_{i,j}(t) := t_i t_j^{-1}, \ i, j \in \mathbb{N}_m \text{ for } i \neq j.$$

The root system

$$\Phi := \{a_{i,j} \mid i, j \in \mathbb{N}_m, \ i \neq j\}$$

of $X^*(T/\mathrm{Z}(T)) \otimes_{\mathbb{Z}} \mathbb{R}$ is of type A_{m-1}. We denote by $u_{i,j}(x)$ the matrix of the homomorphism

$$v_k \mapsto v_k + v_j \delta_{i,k} x$$

The set of k-rational points of $\mathrm{Z}_{\mathbf{GL}_D(V)}(T)$ together with the root groups

$$U_{i,j} := \{u_{i,j}(x) \mid x \in D\}, \ i \neq j, \ i, j \in \mathbb{N}_m,$$

form a valuated root datum using the valuation

$$\phi_{a_{i,j}}(u_{i,j}(x)) := \nu(x). \tag{2.3}$$

For the definition of a valuated root datum see [BT72, 6.2] or B.6. For the example see [BT72, 10]. A short introduction of the steps for the construction of the building of a valuated root datum can be found in the appendix B.

The vector space $W := X_*(T/\mathrm{Z}(T))_k \otimes_{\mathbb{Z}} \mathbb{R}$ is identified with the dual of $X^*(T/\mathrm{Z}(T))_k \otimes_{\mathbb{Z}} \mathbb{R}$ via the natural pairing

$$X^*(T/\mathrm{Z}(T))_k \times X_*(T/\mathrm{Z}(T))_k \to \mathbb{Z}$$

and we denote therefore $X^*(T/\mathrm{Z}(T))_k \otimes_{\mathbb{Z}} \mathbb{R}$ by W^*. The standard apartment is the set Δ of all valuations of

$$(\mathrm{Z}_{\mathrm{GL}_D(V)}(T(k)), (U_{i,j})_{i,j})$$

which are equipollent to ϕ, see section B.2. Δ is an affine space over W. A vector w of W acts on A by

$$\psi \mapsto (u \in U_a \mapsto \psi_a(u) + a(w)).$$

The group $T(k)$ acts on Δ by translation via

$$(t, \psi) \mapsto \psi + w(t)$$

where $w(t) \in W$ is defined by

$$\forall a \in \Phi : \ a(w(t)) = \nu(a(t)).$$

The set $N(T)(k)$ of k-rational pionts of the normaliser $N(T)$ of T in $\tilde{\mathbf{G}}$ precisely consists of the monomial matrixes with entries in D and the above action extends for an element

$n \in N(T)(k)$ via

$$n.(\phi + w) = \phi + n.w$$

where

$$a_{\tau(i)\tau(j)}(n.w) = a_{ij}(w) + \nu(n_{\tau(j),j}) - \nu(n_{\tau(i),i})$$

and τ is the involution defined by $n_{\tau(i),i} \neq 0$.

Definition 2.23 The building $\mathfrak{B}(\tilde{\mathbf{G}}, k)$ of the valuated root datum defined in (2.3) is the set of equivalence classes of $\tilde{G} \times \Delta$ under the relation:

$$(g, x) \sim (h, y)$$

if and only if there exists a monomial matrix n such that

$$n(x) = y \text{ and } hn \in gP_x.$$

The set of all apartments is given by the sets of the form $g\Delta$, $g \in \tilde{G}$ using the $\tilde{G}$-action on the first coordinate. The definition of P_x is given in section B.2 in appendix B. We denote $\mathfrak{B}(\tilde{\mathbf{G}}, k)$ the *Bruhat-Tits building of $\tilde{\mathbf{G}}$ over k*.

Remark 2.24 The Bruhat-Tits building of $\mathbf{SL}_D(V)$ over k is constructed in the same way and it is canonically identified with the Bruhat-Tits building of $\tilde{\mathbf{G}}$ over k.

Remark 2.25 In chapter C we recall the notion of an enlarged building from [BT84a, 4.2.16]. If $\mathfrak{B}(G, k)$ is the Bruhat-Tits building of a reductive group over a local field k we denote the enlarged building by $\mathfrak{B}^1(G, k)$. The group $X^*(\mathbf{SL}_D(V))_k$ is trivial and $X^*(\tilde{\mathbf{G}})_k$ is isomorphic to $\mathbb{Z}$. Thus $\mathfrak{B}^1(\mathbf{SL}_D(V), k)$ and $\mathfrak{B}(\mathbf{SL}_D(V), k)$ coincide and $\tilde{\mathbf{G}}$ has a proper enlarged building over k.

Remark 2.26 The apartments are in one to one correspondence with the maximal k-split tori of $\tilde{\mathbf{G}}$.

As described in [BT84b, 2.11] and [BL02] one can associate to a point x of the enlarged apartment $\Delta^1 := X_*(T)_k \otimes_{\mathbb{Z}} \mathbb{R}$ a D-norm. in the following way.

$$\alpha_x(\sum_i d_i v_i) := \inf_i(\nu(d_i) - a_i(x)) \tag{2.4}$$

where $a_i \in X^*(T)$ is the projection to the ith coordinate and

$$a_i(x) := < x, a_i > .$$

The o_D-lattice function corresponding to α_x is denoted by Λ_x. The maps from Δ^1 to $\mathrm{Norm}_D^1(V)$, $\mathrm{Latt}_{o_D}^1(V)$ resp. induced by (2.4) can be extended to the whole enlarged building and if one asks for some further properties then this extension is possible in a unique way. More precisely by Bruhat, Tits, Broussous and Lemaire we have the

following theorem. Here we use that the dual W^1 of $X_*(\tilde{\mathbf{G}}) \otimes_{\mathbb{Z}} \mathbb{R}$ acts on the enlarged building by translations. For further description see C. One defines an action of W^1 on the set of o_D-lattice functions by

$$\Lambda + \lambda n d \check{a}_0 := \Lambda + \lambda$$

where

$$\check{a}_0 := \frac{1}{n} \sum_i \check{a}_i$$

and where $(\check{a}_i)$ is the dual basis of (a_i) in $X_*(T)$.

Theorem 2.27 *[BL02, I.1.4.,I.2.4, II.1.1. for $F = E$] There is a unique $A^\times$- and W^1-equivariant affine bijection*

$$\mathfrak{B}^1(\tilde{\mathbf{G}}, k) \to \mathrm{Latt}^1_{o_D}(V)$$

extending the map

$$x \in \Delta^1 \mapsto \Lambda_x.$$

This bijection induces the unique $\tilde{G}$-equivariant and affine map from $\mathfrak{B}(\tilde{\mathbf{G}}, k)$ to the set of lattice function classes $\mathrm{Latt}_{o_D}(V)$. It is bijective and an extension of

$$x \in \Delta \mapsto [\Lambda_x].$$

Definition 2.28 The set of k-rational points of the Lie algebra of $\tilde{\mathbf{G}}$ is A. For a point $x \in \mathfrak{B}^1(\tilde{\mathbf{G}}, k)$ we denote the square lattice function corresponding to x by $\mathrm{LF}(x, \tilde{\mathbf{G}}, k)$. This sequence is called the *Lie algebra filtration* of x in A.

   The last proposition of this section is not used in this part of the thesis, but in the next part. We shortly explain the simplicial structure of $\mathfrak{B}(\mathbf{GL}_D(V), k)$. For this we explain the structure for Δ and apply the action of $\mathrm{GL}_D(V)$. The hyperplanes of Δ given by the equations

$$a_{i,j} = \frac{k}{d}$$

for $i, j \in \mathbb{N}_m$ with $i \neq j$ and $k \in \mathbb{Z}$ cut out a cell decomposition of Δ, see for example [BT72, 1.3.3] for the simplicial structure given by an affine root system or [Bro89, VI.1.B] or [Gar97, 12.1]. In the next we consider the last map of the above theorem, i.e. the correspondence with $\mathrm{Latt}_{o_D}(V)$. The ideas of the following proposition are taken form [BT84b, 2.16].

Proposition 2.29 *[BT84b]*

1. *The apartment Δ is mapped to the set of classes of o_D-lattice functions which are split by (v_i).*

2. *An element x of $\mathfrak{B}(\mathbf{GL}_D(V), k)$ lies on a face of rank k if and only if Λ_x has rank k. (We only consider faces which are open in their affine span, i.e. we consider cells.)*

Proof:

1. This follows from the definition.

2. Because of the $\mathrm{SL}_D(V)$-equivariance we only have to consider the subset of Δ given by the inequalities

$$\alpha_{i+1,i}(x) \geq 0 \text{ for } i \in \mathbb{N}_{m-1} \text{ and } \alpha_{m,1}(x) \leq \frac{1}{d}.$$

It is the closure of the chamber C which is defined by the strict inequalities. The image of $\bar{C}$ is the set

$$\{[\Lambda_x] \mid x \in \Delta^1, \ 0 \leq a_1(x) \leq a_2(x) \leq \ldots \leq a_m(x) = \frac{1}{d}\}.$$

The proof now is an easy counting of jumps in the sequence

$$(a_1(x), a_2(x), a_3(x), \ldots, a_{m-1}(x), \frac{1}{d}).$$

q.e.d.

2.3. Self-dual lattice functions

This is a collection of results of [BS09] and [BT87] and we slightly generalise the definition of self-dual objects and propositions of [BS09]. For this section we make the following assumption.

Assumption 2.30 *We fix a local hermitian (k,ν)-datum*

$$((A, V, m, D, d, L|k), \rho, k_0, h, \epsilon, \sigma)$$

and we assume k to have residue characteristic not two (see definition 1.22).

2.3.1. Duality

We explain how h defines a map of order two on all spaces of lattice functions and norms which we have considered. At first we define this map for $\mathrm{Latt}^1_{o_D}(V)$ and then for all sets of the diagrams (2.1) and (2.2).

Definition 2.31 For a lattice function $\Lambda \in \mathrm{Latt}^1_{o_D}(V)$ and a real number r we define

$$\Lambda(r+) := \cup_{s>r}\Lambda(s).$$

Definition 2.32 ([BS09] after Prop. 3.2) Given a lattice $M \in \mathrm{Latt}(V, o_D)$ and a lattice function $\Lambda \in \mathrm{Latt}^1_{o_D}(V)$ the duals are defined by

$$M^{\#} := \{x \in V \mid h(x, M) \subseteq \mathfrak{p}_D\}$$

and
$$\Lambda^{\#}(r) := [\Lambda((-r)+)]^{\#}.$$

Remark 2.33 A D-endomorphism a of V behaves under the dualisation of a lattice in the following way:
$$(a^{\sigma}(M))^{\#} = a^{-1}(M^{\#}).$$

Proposition 2.34   *1. For all $M \in \mathrm{Latt}(V, o_D)$ the set $M^{\#}$ is a full o_D-lattice and $(M^{\#})^{\#} = M$.*

2. *For all $\Lambda \in \mathrm{Latt}^1_{o_D}(V)$ we have $\Lambda^{\#} \in \mathrm{Latt}^1_{o_D}(V)$ and $(\Lambda^{\#})^{\#} = \Lambda$.*

Proof:

1. Let $(v_i)_i$ be a Witt basis of h (see corollary 1.20), and let f be a D-linear automorphism of V which maps
$$M' := \bigoplus_{i=1}^{m} v_i o_D$$
onto M. Assertion 1 is true for M' because the following two equations hold

$$(M')^{\#} = \bigoplus_{i=1}^{m} v_i \mathfrak{p}_D \text{ and } ((M')^{\#})^{\#} = (M'\mathfrak{p}_D)^{\#} = (M')^{\#}\mathfrak{p}_D^{-1} = M'.$$

Further we have
$$f^{\sigma}(M^{\#}) = (M')^{\#}.$$

Therefore $M^{\#}$ is a full lattice and

$$(M^{\#})^{\#} \;=\; (f^{\sigma})^{\sigma}(((M')^{\#})^{\#}) \;=\; f(M') \;=\; M$$

as required.

2. For the first assertion we only show (3) of definition 2.4.

$$\cap_{\epsilon>0}\Lambda^{\#}(r-\epsilon) = \{v \in V \mid h(v, \cup_{\epsilon>0}\Lambda((-r+\epsilon)+)) \subseteq \mathfrak{p}_D\}$$
$$= \{v \in V \mid h(v, \Lambda(-r)+) \subseteq \mathfrak{p}_D\}$$
$$= \Lambda^{\#}(r).$$

The second assertion is true in pairs r, $-r$ of continuity points of Λ because of

$$(\Lambda^{\#})^{\#}(r) = (\Lambda(r)^{\#})^{\#} = \Lambda(r).$$

The density of the set of these r in $\mathbb{R}$ and the left continuity of Λ and $(\Lambda^{\#})^{\#}$ extend the equality to all real nunbers.

q.e.d.

Before we transfer $()^{\#}$ to other spaces, we introduce the analogous definition for the dual of a norm. This was introduced by Bruhat and Tits.

Definition 2.35 The *dual* of a D-norm α on V with respect to h is the D-norm given by
$$\bar{\alpha}(v) := \inf_{w \in V, w \neq 0} (\nu(h(v, w)) - \alpha(w)).$$

We skip the notion "with respect to h" after the following lemma because definition 2.7 is not used after the proof.

Lemma 2.36 [BT87, 2.5] The dual of a norm with respect to h is well defined and if (v_i) is a splitting basis of α and (w_i) is a D-basis of V such that

$$h(w_i, v_j) = \delta_{i,j}$$

then (w_i) is a splitting basis of $\bar{\alpha}$, and the value of $\bar{\alpha}$ in w_i is $-\alpha(v_i)$.

A basis (w_i) as in the above lemma exists, because h is non-degenerate.

Proof: Under
$$(\hat{h})^{\#} : \mathrm{Norm}_D(V^*) \to \mathrm{Abb}(V, \mathbb{R}),$$
$$\beta \mapsto \beta \circ \hat{h},$$

the image of α^* is $\bar{\alpha}$. See proposition 1.12 for the definition of $\hat{h}$. Therefore $\bar{\alpha}$ is an o_D-norm of V. We now prove that (w_i) splits $\bar{\alpha}$. The norm α^* has (v_i^*) as a splitting basis which is the image of (w_i) under $\hat{h}$. Thus (w_i) is a splitting basis of $\bar{\alpha}$, and

$$\bar{\alpha}(w_i) = \alpha^*(v_i) = -\alpha(v_i).$$

q.e.d.

Proposition 2.37 *Under the diagrams (2.1) and (2.2) the map $()^{\#}$ corresponds to the following maps:*

1. *on* $\mathrm{Norm}_D^1(V) : \alpha \mapsto \bar{\alpha}$

2. *on* $\mathrm{Norm}_D(V) : [\alpha]^{\sigma} := [\bar{\alpha}],$

3. *on* $\mathrm{Latt}_{o_D}(V) : [\Lambda]^{\sigma} := [\Lambda^{\#}].$

4. *on* $\mathrm{Latt}_{o_k}^2(A) : \mathfrak{a}^{\sigma}(t) := (\mathfrak{a}(t))^{\sigma},$

5. *on* $\mathrm{Norm}_k^2(A) : \beta^{\sigma}(f) := \beta(f^{\sigma}),$

Proof: 1. The proof is similar to [BS09, 3.3.].
2. and 3. The maps are well defined which follows immediately from the definitions.

4. We prove for all $\Lambda \in \text{Latt}^1_{o_D}(V)$:

$$\text{End}(\Lambda^{\#}) = \text{End}(\Lambda)^{\sigma}.$$

For an element a of A the following statements are equivalent.

- $a^{\sigma} \in \text{End}(\Lambda^{\#})(r)$

- The lattice $a^{\sigma}([\Lambda((-s)+)]^{\#})$ is contained in $[\Lambda((-s-r)+)]^{\#}$ for all real numbers s.

- The lattice $a\Lambda((-s-r)+)$ is contained in $\Lambda((-s)+)$ for all real numbers s by remark 2.33.

- a is an element of $\text{End}(\Lambda)(r)$.

5. The bijection

$$\text{Norm}^1_k(A) \cong \text{Latt}^1_{o_k}(A)$$

maps the norm $\text{End}(\alpha) \circ \sigma$ to $\text{End}(\Lambda_{\alpha})^{\sigma}$ which is $\text{End}(\Lambda^{\#}_{\alpha})$ by assertion 4. The commutativity of diagram (2.1) implies that $\text{End}(\bar{\alpha})$ is mapped to $\text{End}(\Lambda_{\bar{\alpha}})$ which is $\text{End}(\Lambda^{\#}_{\alpha})$ by assertion 1. The equality

$$\text{End}(\alpha) \circ \sigma = \text{End}(\bar{\alpha})$$

follows now from the injectivety of the above bijection. q.e.d.

Remark 2.38   1. All maps given in 2.37 have order two by proposition 2.34.

2. Let Λ be an o_D-lattice function. For $g \in \text{GL}_D V$ we have

$$(g\Lambda)^{\#} = (g^{\sigma})^{-1} \Lambda^{\#}.$$

Proposition 2.39 *The map*

$$()^{\#} : \text{Latt}^1_{o_D}(V) \rightarrow \text{Latt}^1_{o_D}(V)$$

is affine and $\text{U}(h)$*-equivariant.*

Proof: The equivariance follows from 2.38[2.]. We prove the affineness with norms and lemma 2.36. Let (v_i) be a splitting basis of two D-norms α and α'. We choose an element $\lambda \in [0,1]$. The basis (v_i) also splits $\gamma := \lambda\alpha + (1-\lambda)\alpha'$. By lemma 2.36 the D-basis (w_i) fulfilling

$$h(w_i, v_j) = \delta_{i,j}$$

splits $\bar{\alpha}$, $\bar{\alpha}'$ and $\bar{\gamma}$ and the values at w_i are

$$\bar{\alpha}(w_i) = -\alpha(v_i), \ \bar{\alpha}'(w_i) = -\alpha'(v_i)$$

and

$$\bar{\gamma}(w_i) = -\gamma(v_i) = -\lambda\alpha(v_i) - (1-\lambda)\alpha'(v_i).$$

Thus

$$\bar{\gamma}(w_i) = \lambda\bar{\alpha}(w_i) + (1-\lambda)\bar{\alpha'}(w_i),$$

which proves the affiness of the map $\alpha \mapsto \bar{\alpha}$. q.e.d.

2.3.2. MM-norms

We are interested in the sets of self-dual objects. For the self-dual norms Bruhat and Tits gave another definition, the definition of an MM-norm.

Definition 2.40 ([BT87][2.1)] One says that $\alpha \in \mathrm{Norm}^1_D(V)$ is *dominated by h*, if for all $v, v' \in V$ we have

$$\alpha(v) + \alpha(v') \le \nu(h(v, v')). \tag{2.5}$$

Remark 2.41 If $(v_i)_i$ is a splitting basis for α then α is dominated by h if and only if for all i, j we have

$$\alpha(v_i) + \alpha(v_j) \le \nu(h(v_i, v_j)).$$

We make $\mathrm{Norm}^1_D(V)$ to a poset by defining $\alpha \le \beta$ if $\alpha(v) \le \beta(v)$ for all $v \in V$.

Definition 2.42 ([BT87] 2.1) A maximal element of the set of $\alpha \in \mathrm{Norm}^1_D(V)$ dominated by h is called a *MM-norm for h* (maximinorante in French).

Lemma 2.43 A D-norm α satisfies the following three properties.

1. For all $v, v' \in V$ we have

$$\alpha(v) + \bar{\alpha}(v') \le \nu(h(v, v')). \tag{2.6}$$

2. $bary(\alpha) := \frac{1}{2}\alpha + \frac{1}{2}\bar{\alpha}$ is dominated by h.

3. If α is dominated by h then $bary(\alpha) \ge \alpha$.

Proof: The first assertion follows from the definition of $\bar{\alpha}$ and it implies the second assertion because $\overline{bary(\alpha)} = bary(\alpha)$ Point 3 follows because to be dominated by h is equivalent to $\alpha \le \bar{\alpha}$. q.e.d.

A part of [BT87, 2.5] is the following proposition.

Proposition 2.44 (F. Bruhat, J. Tits) *For $\alpha \in \mathrm{Norm}^1_D(V)$ the following statements are equivalent.*

1. $\alpha = \bar{\alpha}$.

2. α is a MM-norm.

Proof: 1.⇒2. : α is dominated by h, since $\alpha \le \bar{\alpha}$. If $\gamma \ge \alpha$ and γ is dominated by h, then $\gamma \le \bar{\gamma} \le \bar{\alpha} = \alpha$, thus 2. 2.⇒1. : By remark 2.43 (2 and 3) we get $\alpha = bary(\alpha)$. Thus $\alpha = \bar{\alpha}$. q.e.d.

2.3.3. Self-duality

We obtain two diagrams with sets of self-dual objects from the diagrams (2.1) and (2.2). An element of one of the sets given in these diagrams is called *self-dual* if it is a fixed point of the corresponding map given in proposition 2.37.

Notation 2.45 We denote the set of self-dual objects as follows:

- for $\mathrm{Norm}_D^1(V)$, $\mathrm{Norm}_D(V) : \mathrm{Norm}_h^1(V)$, $\mathrm{Norm}_h(V)$,

- for $\mathrm{Latt}_{o_D}^1(V)$, $\mathrm{Latt}_{o_D}(V) : \mathrm{Latt}_h^1(V)$, $\mathrm{Latt}_h(V)$

- for $\mathrm{Norm}_k^2(A) : \mathrm{Norm}_\sigma^2(A)$,

- for $\mathrm{Latt}_{o_k}^2(A) : \mathrm{Latt}_\sigma^2(A)$.

The next proposition is a corollary of proposition 2.39.

Proposition 2.46 *The sets* $\mathrm{Norm}_h^1(V)$ *and* $\mathrm{Latt}_h^1(V)$ *are closed under the affine structure of* $\mathrm{Norm}_h^1(V)$ *and* $\mathrm{Latt}_h^1(V)$ *respectively.*

Proposition 2.47 *We get two commutative diagrams of* $\mathrm{U}(h)$*-equivariant maps.*

$$
\begin{array}{ccc}
\mathrm{Norm}_h(V) & \rightarrow & \mathrm{Latt}_h(V) \\
\downarrow & & \downarrow \\
\mathrm{Norm}_\sigma^2(A) & \rightarrow & \mathrm{Latt}_\sigma^2(A)
\end{array}
\tag{2.7}
$$

$$
\begin{array}{ccc}
\mathrm{Norm}_h^1(V) & \rightarrow & \mathrm{Latt}_h^1(V) \\
\downarrow & & \downarrow \\
\mathrm{Norm}_h(V) & \rightarrow & \mathrm{Latt}_h(V)
\end{array}
\tag{2.8}
$$

The maps in the second diagram are affine.

The vertical maps of diagram (2.8) are surjective. Indead if a D-norm α satisfies $[\alpha]^\sigma = [\alpha]$ there is a real number s such that

$$\bar\alpha = \alpha + s.$$

It follows that the norm $\alpha + \frac{s}{2}$ lies in $\mathrm{Norm}_h^1(V)$, because

$$\overline{\alpha + \frac{s}{2}} = \bar\alpha - \frac{s}{2} = \alpha + \frac{s}{2}.$$

Remark 2.48 An analogous argument shows that two self-dual D-norms are equivalent if and only if they equal, i.e. that all maps of diagram (2.8) are bijective.

We consider the direct sum of self-dual lattice functions.

Remark 2.49 Let V' be another finite dimensional D-right vector space with an ϵ-hermitian form h'. On $V \oplus V'$ we have the ϵ-hermitian form $\tilde{h}$ defined by

$$\tilde{h}((v, v'), (w, w')) = h(v, w) + h'(v', w').$$

1. We have

$$(\Lambda \oplus \Lambda')^{\#} = \Lambda^{\#} \oplus \Lambda'^{\#}$$

for two lattice functions $\Lambda \in \mathrm{Latt}^1_{o_D}(V)$ and $\Lambda' \in \mathrm{Latt}^1_{o_D}(V')$.

2. The direct sum of self-dual lattice functions is self-dual.

Assumption 2.50 *For the last part of this subsection let us assume that h is isotropic and that*

$$V = W \oplus W'$$

with maximal totally isotropic subspaces of V. We put $k := \dim_D W$.

Definition 2.51 For $M \in \mathrm{Latt}(W, o_k)$ we define *its dual in W'* by

$$M^{\#,W'} := \{w' \in W' \mid h(w', M) \subseteq \mathfrak{p}_D\},$$

and analogously $M'^{\#,W}$ for $M' \in \mathrm{Latt}(W', o_k)$. The *dual of $\Lambda \in \mathrm{Latt}^1_{o_D}(W)$ in W'* is defined by

$$\Lambda^{\#,W}(t) := (\Lambda((-t)+))^{\#,W'},$$

and we have an anologous definition for o_D-lattice functions of W'.

Proposition 2.52 *For $M \in \mathrm{Latt}(W, o_k)$, $Q \in \mathrm{Latt}(W', o_k)$, $\Lambda \in \mathrm{Latt}^1_{o_D}(W)$ and $\Lambda' \in \mathrm{Latt}^1_{o_D}(W')$ we have:*

1. *$(M \oplus Q)^{\#} = Q^{\#,W} \oplus M^{\#,W'}$ and $Q^{\#,W}$ and $M^{\#,W'}$ are full lattices in the corresponding vector spaces.*

2. *$(\Lambda \oplus \Lambda')^{\#} = \Lambda'^{\#,W} \oplus \Lambda^{\#,W'}$ and $\Lambda'^{\#,W}$ and $\Lambda^{\#,W'}$ are lattice functions in the corresponding vector spaces.*

Proof: For 1.: The equality is a consequence of $h(W, W) = h(W', W') = \{0\}$, i.e.

$$(w, w') \in (M \oplus Q)^{\#}$$

if and only if

$$h(w, Q) + h(M, w') \subseteq \mathfrak{p}_D$$

if and only if

$$h(w, Q) \cup h(M, w') \subseteq \mathfrak{p}_D$$

if and only if

$$w \in Q^{\#,W} \text{ and } w' \in M^{\#,W'}.$$

The set $(M \oplus Q)^{\#}$ is a full lattice in V and by the equality we get that $Q^{\#,W}$ is a full lattice in W and $M^{\#,W'}$ is a full lattice in W'.

For 2.: From 1. we get the equality. The left side is a lattice function. Thus both summands on the right side of the equation are lattice functions. q.e.d.

Proposition 2.53   *1. The maps*

$$()^{\#,W'} : \mathrm{Latt}(W, o_k) \rightarrow \mathrm{Latt}(W', o_k)$$

and

$$()^{\#,W} : \mathrm{Latt}(W', o_k) \rightarrow \mathrm{Latt}(W, o_k)$$

are inverse to each other.

2. The maps from 1 are affine.

Proof:

1. We take $M \in \mathrm{Latt}(W, o_k)$ and $Q \in \mathrm{Latt}(W', o_k)$ and by 1. of proposition 2.52 we get

$$((M \oplus Q)^{\#})^{\#} = (Q^{\#,W} \oplus M^{\#,W'})^{\#} = (M^{\#,W'})^{\#,W} \oplus (Q^{\#,W})^{\#,W'}.$$

From 2. of proposition 2.34 we get

$$((M \oplus Q)^{\#})^{\#} = M \oplus Q.$$

Both equalities together imply the first assertion.

2. By proposition 2.39 the map $()^{\#}$ on $\mathrm{Latt}^1_{o_D}(V)$ is affine. By remark 2.12 and 2.52[2.] we get the second assertion.

q.e.d.

Definition 2.54 For an endomorphism $a \in \mathrm{End}_D(W)$, there is a unique endomorphism of W' denoted by $a^{\sigma,W'}$ such that $(a \oplus 0)^{\sigma} = 0 \oplus a^{\sigma,W'}$. We define an embedding

$$i_{W,W'} : \mathrm{GL}_D(W) \rightarrow \mathrm{U}(h)$$

as follows

$$i_{W,W'}(g)(w, w') := (g(w), (g^{\sigma,W'})^{-1}(w')).$$

The map $i_{W,W'}$ defines a k-morphism and the differential at identity is given by

$$di_{W,W'}(a) := a \oplus (-a^{\sigma,W'}).$$

Its image is a subset of $\mathrm{Lie}(\mathbf{U}(h))(k_0)$.

Proposition 2.55 *The map*

$$\phi : \mathrm{Latt}^1_{o_D}(W) \to \mathrm{Latt}^1_h(V)$$

defined by

$$\phi(\Lambda) := \Lambda \oplus \Lambda^{\#,W'}$$

is affine and $GL_D(W)$-*equivariant, i.e.*

$$\phi(g\Lambda) = i_{W,W'}(g)\phi(\Lambda).$$

Proof: We have $\Lambda \oplus \Lambda^{\#,W'} \in \mathrm{Latt}^1_h(V)$ by 2.52[2.] and 2.53[1.]. The affineness follows from 2.53[2.] and remark 2.12. For the equivariance we need

$$(g\Lambda)^{\#,W'} = (g^{\sigma,W'})^{-1}\Lambda^{\#,W'}$$

which follows from

$$h((0,w'),(g(w),0)) = h((0,g^{\sigma,W'}(w')),(w,0)).$$

q.e.d.

2.4. The Bruhat-Tits building of $\mathbf{U}(h)$

We adopt assumption 2.30.

Remark 2.56 From 1.37 we deduce that D can only have an index d which is 1 or 2, and if the index is 2 then k_0 equals k. Without loss of generality we can assume $\epsilon = -1$ if $d = 2$ by [BT87, (22.a)].

In this section we describe the Bruhat-Tits building of $\mathbf{SU}(h)$ as a subset of $\mathfrak{B}^1(\tilde{\mathbf{G}},k)$. It was done by Bruhat and Tits in terms of norms. We use the concept of self dual lattice functions from the last section. This description was introduced in [BS09] based on [BT87].

Notation 2.57 We denote by $\mathbf{O}^{is}_{2,k}$ the k-split isotropic orthogonal group of rank 1, i.e. the unitary group given by an isotropic symmetric k-bilinear form on k^2.

Example 2.58 The connected component of $\mathbf{O}^{is}_{2,k}$ is k-isomorphic to $\mathbf{G_m}$ and we can apply section 2.2. Its Bruhat-Tits building over k is a point and its enlarged building is a line. The $\mathbf{G_m}(k)$-action on $\mathrm{Latt}^1_{o_k}(k)$ is extended to an $\mathbf{O}^{is}_{2,k}(k)$-action via

$$\mathrm{antidiag}(1,1).(s \mapsto \mathfrak{p}^{[s-y]+}_k) := (s \mapsto \mathfrak{p}^{[s+y]+}_k).$$

Proposition 2.59 *The following three assertions are equivalent.*

1. *There is a k_0-isomorphism between* $\mathbf{U}(h)$ *and* $\mathbf{O}^{is}_{2,k_0}$.

2. *There is a k_0-isomorphism between* $\mathbf{SU}(h)$ *and* $\mathbf{G_m}$.

3. *The following system of conditions is satisfied:*

$$D = k = k_0 \ \text{and} \ m = 2 \ \text{and} \ \sigma \ \text{is orthogonal}$$

$$\text{and} \ h \ \text{is isotropic.}$$

Proof: The implication $3\Rightarrow1$ and $1\Rightarrow2$ are obvious and $2\Rightarrow3$ is a direct consequence of C.7. q.e.d.

Assumption 2.60 *In the following introduction we assume that* $\mathbf{U}(h)$ *is not k_0-isomorphic to* $\mathbf{O}_{2,k_0}^{is}$.

We consider the building of the valuated root datum given in [BT87, 1.15.], denote it by $\mathfrak{B}(\mathbf{SU}(h), k_0)$ and call it the *Bruhat-Tits building of* $\mathbf{SU}(h)$ *over* k_0.

Remark 2.61 1. From the definition of a building corresponding to a valuated root datum it follows that in the anisotropic case the building is a point and in the isotropic case the building is the geometric realisation of a thick Euclidean building.

2. The apartments are in one to one correspondence with the Witt decompositions.

Let us now fix a Witt basis $(v_i)_{I \cup I_0}$ of V with respect to h, i.e. we have the Witt decomposition

$$V_i := v_i D, \ i \in I, \ V_0 := \sum_{i \in I_0} v_i D.$$

Let T be the torus defined over k_0 whose set of k_0-rational points is given by the k_0-rational points t of $\mathbf{SU}(h)$ satisfying:

1. $t.v_i \in k_0 v_i$ for all $i \in I$ and

2. $t.v = v$ for all $v \in V_0$

Then T is a maximal k_0-split torus of $\mathbf{SU}(h)$. We look at the characters a_i defined on $T(K_0)$ by

$$t v_i = \alpha_i(t)^{-1} v_i, \ t \in T(K_0).$$

There is a bijection from Δ the apartment of $\mathfrak{B}(\mathbf{SU}(h), k_0)$ corresponding to the torus T to the set of the MM-norms which split under the given Witt basis.

$$x \in \Delta \mapsto \alpha_x$$

where

$$\alpha_x(\sum_{i \in I} v_i \lambda_i + v_0) := \inf\{\frac{1}{2}\nu(q(v_0)), \inf_{i \in I}\{\nu(\lambda_i) - a_i(x)\}\}.$$

Here q is the pseudo-quadratic form corresponding to h. (see [BT87, 1.2. (9)]) This gives a map from Δ to $\mathrm{Latt}_h^1(V)$, whose image is the set of self-dual lattice functions which are split under our Witt basis. This set is denoted by $\mathrm{Latt}_{h,(V_i)_i}^1(V)$. We denote the self dual lattice function corresponding to $x \in \Delta$ by Λ_x.

Definition 2.62 If $\mathbf{U}(h)$ is connected then the Bruhat-Tits building $\mathfrak{B}(\mathbf{U}(h), k_0)$ of $\mathbf{U}(h)$ is defined analogously and canonically identifies with $\mathfrak{B}(\mathbf{SU}(h), k_0)$. If $\mathbf{U}(h)$ is not connected we define $\mathfrak{B}(\mathbf{U}(h), k_0)$ to be $\mathfrak{B}(\mathbf{SU}(h), k_0)$.

Remark 2.63 For the case we consider there is no proper enlarged building of $\mathbf{SU}(h)$ and $\mathbf{U}(h)$ because $X^*(\mathbf{SU}(h))_{k_0}$ and $X^*(\mathbf{U}(h)^0)_{k_0}$ are trivial, i.e. we have

$$\mathfrak{B}(\mathbf{SU}(h), k_0) = \mathfrak{B}^1(\mathbf{SU}(h), k_0) = \mathfrak{B}(\mathbf{U}(h), k_0) = \mathfrak{B}^1(\mathbf{U}(h), k_0).$$

Broussous and Stevens proved a reformulation of [BT87, 2.12] for the case where $D = k$. With minor changes their proof is valid for the case $D \neq k$ (see also [Lem09] §4).

Theorem 2.64 *[BS09, Prop. 4.2.] There a unique* $\mathbf{U}(h)(k_0)$-*equivariant affine map*

$$\mathfrak{B}(\mathbf{U}(h), k_0) \rightarrow \mathrm{Latt}_h^1(V).$$

It is bijective and an extension of the map

$$x \in \Delta \mapsto \Lambda_x.$$

Remark 2.65 In the omitted case, see example 2.58, we have

$$\mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k) \cong \mathrm{Latt}_{o_k}^1(V) \cong \mathbb{R}, \ (\mathfrak{p}_k^{[s-y]+})_{s \in \mathbb{R}} \mapsto y.$$

The affine space $\mathrm{Latt}_h^1(V)$ can also be identified with $\mathbb{R}$ if one fixes a Witt-basis (v_1, v_2) for the unique Witt decomposition of V, precisely

$$y \in \mathbb{R} \mapsto \ (\mathfrak{p}_k^{[s-y]+} v_1 \oplus \mathfrak{p}_k^{[s+y]+} v_2)_s.$$

The identity of $\mathbb{R}$ induces the unique $\mathbf{O}_{2,k}^{is}(k)$-equivariant affine bijection from $\mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k)$ to $\mathrm{Latt}_h^1(V)$ because the identity is the only affine map j of $\mathbb{R}$ which satisfies $j(y+1) = j(y) + 1$ and $j(-y) = -j(y)$ for all $y \in \mathbb{R}$.

We also have a notion of a Lie algebra filtration here.

Definition 2.66 The set of k_0-rational points of $\mathrm{Lie}(\mathbf{U}(h))$ is the set

$$\{A \in a \mid a + \sigma(a) = 0\}$$

of skewsymmetric D-endomorphisms of V with respect to σ. For a point x of $\mathfrak{B}^1(\mathbf{U}(h), k_0)$ the following intersection

$$\mathrm{LF}(x, \mathbf{U}(h), k_0) := \mathrm{LF}(x, \tilde{\mathbf{G}}, k) \cap \mathrm{Lie}(\mathbf{U}(h))(k_0)$$

defines an o_{k_0}-lattice function in $\mathrm{Lie}(\mathbf{U}(h))(k_0)$. It is called the *Lie algebra filtration* of x in $\mathrm{Lie}(\mathbf{U}(h))(k_0)$.

Theorem 2.67 *[Lem09] The filtration* $\mathrm{LF}(x, \mathbf{U}(h), k_0)$ *coincides with the Moy-Prasad filtration.*

3. Maps which are compatible with the Lie algebra filtrations

3.1. Compatibility with the Lie algebra filtrations

The notion of CLF-map was introduced in [BL02]. Let F be non-Archimedean local field with valuation ring o_F.

Definition 3.1 Let B be a finite dimensional Lie algebra over F. An o_F-Lie algebra filtration of B is an o_F-lattice function of B.

Definition 3.2 1. A reductive F-group G with the Bruhat-Tits building $\mathfrak{B}(G, F)$ is said to be an *LF-F-group*, if every point x of $\mathfrak{B}(G, F)$ is attached to an o_F-Lie algebra filtration $\mathrm{LF}(x, G, F)$ of $\mathrm{Lie}(G)(F)$. If there is no confusion we skip the prefix o_F.

  2. The Lie algebra filtration of a Lie algebra B attached to a point x is also denoted by $\mathrm{LF}(x, B)$.

If G is a connected LF-F-group we can attach a Lie algebra filtration to every element of the enlarged building $\mathfrak{B}^1(G, k)$ if we use the projection to the first component

$$\mathfrak{B}^1(G, F) \rightarrow \mathfrak{B}(G, F), \ (y, w) \mapsto y,$$

see C, i.e. we define

$$\mathrm{LF}((y, w), \mathbf{G}, F) := \mathrm{LF}(y, \mathbf{G}, F).$$

Definition 3.3 Let $\mathbf{H}$ and $\mathbf{G}$ be LF-F-groups. Let $i : \mathbf{H} \rightarrow \mathbf{G}$ be an F-homomorphism. We call a point y of $\mathfrak{B}(\mathbf{G}, F)$ an *extension of $x \in \mathfrak{B}(\mathbf{H}, F)$ with respect to i* if

$$\mathrm{LF}(y, \mathbf{G}, F) \cap \mathrm{im}(di) = di(\mathrm{LF}(x, \mathbf{H}, F)) \tag{3.1}$$

where $di : \mathrm{Lie}(\mathbf{H}) \rightarrow \mathrm{Lie}(\mathbf{G})$ is the differential of i. We omit to mention i if the choice of i is clear. An analogous definition can be made using also enlarged buildings.

Definition 3.4 Under the assumptions of definition 3.3 a map j between subsets of the buildings $\mathfrak{B}(\mathbf{H}, F)$ and $\mathfrak{B}(\mathbf{G}, F)$ is *compatible with the Lie algebra filtrations (CLF) with respect to i* if we have that an element y of $\mathfrak{B}(\mathbf{G}, F)$ is an extension of $x \in \mathfrak{B}(\mathbf{H}, F)$ if j maps x to y or y to x. We give analogous definitions for maps between subsets of enlarged buildings or between subsets of an enlarged and a non-enlarged building.

Example 3.5 Assume we have given a local hermitian k-datum of residue characteristic not two with unitary involution σ. We have the inclusion

$$\mathbf{U}(h) \to \operatorname{Res}_{k|k_0}(\mathbf{GL}_k(V)) \overset{/k}{\cong} \mathbf{GL}_k(V) \times \mathbf{GL}_k(V),$$

$$\operatorname{U}(h) = \mathbf{U}(h)(k_0) \subseteq \operatorname{Res}_{k|k_0}(\mathbf{GL}_k(V))(k_0) = \operatorname{GL}_k(V) = \mathbf{GL}_k(V)(k)$$

and

$$\operatorname{Lie}(\mathbf{U}(h))(k_0) \subseteq \operatorname{Lie}(\operatorname{Res}_{k|k_0}(\mathbf{GL}_k(V)))(k_0) = \operatorname{End}_k(V) = \operatorname{Lie}(\mathbf{GL}_k(V))(k).$$

Thus we have a notion of CLF for maps

$$\mathfrak{B}^1(\mathbf{U}(h), k_0) \to \mathfrak{B}^1(\mathbf{GL}_k(V), k) = \mathfrak{B}^1(\operatorname{Res}_{k|k_0}(\mathbf{GL}_k(V)), k_0).$$

The aim of this work is to analyse how precise a map is determined by the CLF property.

3.2. Buildings of centralisers

Assumption 3.6 *For the rest of part 1 we adopt assumption 2.1 and we assume that k has residue characteristic not two.*

From section 3.2 to section 4.5 we only consider centralisers of **separable** Lie algebra elements and separable field extensions. In section 4.6 we explain how the results of chapter 3 and 4 generalise to the non-separable case.

Notation 3.7 For a group action

$$G \times W \to W$$

we denote the fixator of an element $w \in W$ by G_w and of a subset S of W by G_S. If W is the Lie algebra of an algebraic group and if we do not specify the action, we use the adjoint group action.

3.2.1. The case of $\operatorname{GL}_D(V)$

Let E be a commutative separable k-subalgebra of $\operatorname{Lie}(\tilde{\mathbf{G}})(k)$, i.e. E splits into a product of separable field extensions of k:

$$E = \prod_i E_i.$$

If 1_i is the idempotent corresponding to 1_{E_i} we put $V_i := 1_i V$. For every i there is an E_i-algebra isomorphism

$$\operatorname{End}_{E_i \otimes_k D}(V_i) \cong \operatorname{End}_{\Delta_i}(W_i)$$

for some skewfields Δ_i central over E_i and some finite dimensional Δ_i-vector spaces W_i.

By the separability of E over k there is a canonical k-isomorphism

$$\tilde{\mathbf{G}}_E \cong \prod_i \mathbf{GL}_D(V_i)_{E_i}$$

and thus the centraliser $\tilde{\mathbf{G}}_E$ is k-isomorphic to

$$\prod_i \operatorname{Res}_{E_i|k}(\mathbf{GL}_{\Delta_i}(W_i)).$$

The building of $\tilde{\mathbf{G}}_E$ over k is $\mathrm{GL}_D(V)_E$-equivariantly isomorphic to

$$\prod_i \mathfrak{B}(\mathbf{GL}_{\Delta_i}(W_i), E_i) \tag{3.2}$$

and the enlarged building is $\mathrm{GL}_D(V)_E$-equivariantly isomorphic to

$$\prod_i \mathfrak{B}^1(\mathbf{GL}_{\Delta_i}(W_i), E_i). \tag{3.3}$$

We identify these products with $\mathfrak{B}(\tilde{\mathbf{G}}_E, k)$ and $\mathfrak{B}^1(\tilde{\mathbf{G}}_E, k)$ respectively, and we work with the lattice function models of the factors.

Notation 3.8 Instead of $\mathbf{GL}_{\Delta_i}(W_i)$ we write $\mathbf{GL}_{E_i \otimes_k D}(V_i)$.

Definition 3.9 The Lie algebra filtration of a point $x = (x_i)$ of $\mathfrak{B}(\tilde{\mathbf{G}}_E, k)$ or the enlarged building $\mathfrak{B}^1(\tilde{\mathbf{G}}_E, k)$ is given by the direct sum of the Lie algebra filtrations of the points x_i, i.e.

$$\mathrm{LF}(x, \tilde{\mathbf{G}}_E, k)(t) := \oplus_i \mathrm{LF}(x_i, \mathbf{GL}_{E_i \otimes_k D}(V_i), E_i)(t), \ t \in \mathbb{R},$$

the sum of the corresponding square lattice functions.

3.2.2. The case of $\mathbf{U}(h)$

Here we use the same idea as in the previous subsection. We take a local hermitian datum with the fixed simple datum of assumption 3.6. We consider the unitary group $\mathbf{G} := \mathbf{U}(h) \subseteq \operatorname{Res}_{k|k_0}(\tilde{\mathbf{G}})$

Let β be an element of

$$\mathrm{Lie}(\mathbf{G})(k_0) = \{a \in \operatorname{End}_D(V) | \ a^\sigma + a = 0\}$$

which is **separable** over k, and we put $\mathbf{H} := \mathbf{G}_\beta$.

The semisimplicity of $k[\beta]$ gives us the following decompositions:

- $E := k[\beta] = \prod_{i \in J} E_i$, a product of fields,

- $1 = \sum_{i \in J} 1_i$, the decomposition of 1 into primitive idempotents,

3. Maps which are compatible with the Lie algebra filtrations

- $V := \oplus_{i \in J}(V_i)$, $V_i := 1_i V$ and

- $\beta = \sum_{i \in J} \beta_i$, $\beta_i = 1_i \beta$.

On J we choose a representation system J_{un+} for the equivalence relation defined by

$$i \sim j \text{ if } i = j \text{ or } \sigma(1_i) = 1_j$$

and we put

$$J_{un} := \{j \in J \mid \sigma(1_j) = 1_j\} \quad J_+ := J_{un+} \setminus J_{un}$$

and

$$J_- := J \setminus J_{un+}.$$

We define $-i := j$ if $\sigma(1_i) = 1_j$. For $i \in J_{un}$ we denote $(E_i)_0$ to be the set of fixed points of σ in E_i.

We obtain a lattice function model for the enlarged building of $\mathbf{H}$ in the following way. The following polynomial isomorphism

$$
\begin{aligned}
\mathbf{H}(k_0) = \mathbf{G}(k_0)_\beta \\
= \prod_{i \in J_{un+}} U(h_{(V_i+V_{-i}) \times (V_i+V_{-i})})_{\beta_i+\beta_{-i}} \\
\cong \prod_{i \in J_{un}} U(h_{V_i \times V_i})_{\beta_i} \times \prod_{i>0} \mathrm{GL}_{E_i \otimes_k D}(V_i)_{\beta_i} \\
= \prod_{i \in J_{un}} \mathrm{Res}_{(E_i)_0|k_0}(\mathbf{U}(\sigma|_{\mathrm{End}_{E_i \otimes_k D}(V_i)}))(k_0) \times \prod_{i>0} \mathrm{Res}_{E_i|k_0}(\mathbf{GL}_{E_i \otimes_k D}(V_i))(k_0)
\end{aligned}
$$

extends to an algebraic isomorphism

$$
\mathbf{H} \cong \prod_{i \in J_{un}} \mathrm{Res}_{(E_i)_0|k_0}(\mathbf{U}(\sigma|_{\mathrm{End}_{E_i \otimes_k D}(V_i)})) \times \prod_{i>0} \mathrm{Res}_{E_i|k_0}(\mathbf{GL}_{E_i \otimes_k D}(V_i))
$$

because $\mathbf{H}(k_0)$ is Zariski-dense in $\mathbf{H}$ because $\mathbf{H}$ is reductive and defined over the infinite field k_0 by the separability of β. For he definition of $\mathbf{U}(\sigma|_{\mathrm{End}_{E_i \otimes_k D}(V_i)})$ see 1.35. For a reductive group Γ defined over a local field L, we have

$$\mathfrak{B}^1(\Gamma, L) \cong \mathfrak{B}^1(\mathrm{Res}_{L|F}(\Gamma), F)$$

for every finite separable field extension $L|F$. Thus $\mathfrak{B}^1(\mathbf{H}, k_0)$ is isomorphic to

$$\prod_{i \in J_{un}} \mathfrak{B}^1(\mathrm{Res}_{(E_i)_0|k_0}(\mathbf{U}(\sigma|_{\mathrm{End}_{E_i \otimes_k D}(V_i)})), k_0) \times \prod_{i>0} \mathfrak{B}^1(\mathrm{Res}_{E_i|k_0}(\mathbf{GL}_{E_i \otimes_k D}(V_i)), k_0)$$

which by the method of restriction of scalars is isomorphic to

$$\cong \prod_{i \in J_{un}} \mathfrak{B}^1(\mathbf{U}(\sigma|_{\mathrm{End}_{E_i \otimes_k D}(V_i)}), (E_i)_0) \times \prod_{i>0} \mathfrak{B}^1(\mathbf{GL}_{E_i \otimes_k D}(V_i), E_i).$$

Definition 3.10 Analogously to definition 3.9 the Lie algebra filtration of a point $x = (x_i)$ of $\mathfrak{B}^1(\mathbf{H}, k_0)$ is defined to be the direct sum of the Lie algebra filtrations of the points x_i, i.e. for $t \in \mathbb{R}$ we put $\mathrm{LF}(x, \mathbf{H}, k_0)(t)$ to be

$$\prod_{i \in J_{un}} \mathrm{LF}(x_i, \mathrm{Skew}(\mathrm{End}_{E_i \otimes_k D}(V_i), \sigma))(t) \times \prod_{i \in J_+} \mathrm{LF}(x_i, \mathbf{GL}_{E_i \otimes_k D}(V_i), E_i)(t).$$

3.2.3. Notation and simplification for the unitary case

We use the notation of subsection 3.2.2. Under the choice of J_+ we make the following simplification of the situation. We put $J_{GL} := J_+ \cup J_-$ and we introduce the following notation. For a symbol $a \in \{+, -, un, GL\}$ we put

$$V_a := \sum_{i \in J_a} (V_i), \ E_a := \sum_{i \in J_a} E_i, \ 1_a := \sum_{i \in J_a} 1_i, \ \beta_a := \sum_{i \in J_a} \beta_i, \ \tilde{\mathbf{G}}_a := \mathbf{GL}_D(V_a),$$

for $b \in \{un, GL\}$ we put

$$h_b := h|_{V_b \times V_b}, \ \sigma_b := \sigma|_{\mathrm{End}_D(V_b)}, \ \mathbf{G}_b := \mathbf{U}(h_b), \ \mathbf{H}_b := (\mathbf{G}_b)_{\beta_b}$$

and for $c \in \{un, +\}$ we put $\tilde{\mathbf{H}}_c := (\tilde{\mathbf{G}}_c)_{E_c}$. For example we have the following commutative diagrams.

$$\begin{array}{ccc}
\tilde{\mathbf{G}}_{GL}(k) \times \tilde{\mathbf{G}}_{un}(k) & \to & \tilde{\mathbf{G}}(k) \\
\uparrow & & \uparrow \\
\mathbf{G}_{GL}(k_0) \times \mathbf{G}_{un}(k_0) & \to & \mathbf{G}(k_0),
\end{array}$$

$$\begin{array}{ccc}
\tilde{\mathbf{H}}_{un}(k) & \to & \tilde{\mathbf{G}}_{un}(k) \\
\uparrow & & \uparrow \\
\mathbf{H}_{un}(k_0) & \to & \mathbf{G}_{un}(k_0),
\end{array}$$

Remark 3.11 We have

$$\mathfrak{B}^1(\mathbf{H}, k_0) = \mathfrak{B}^1(\mathbf{H}_{un}, k_0) \times \mathfrak{B}^1(\mathbf{H}_{GL}, k_0).$$

3.3. CLF-maps in the case of $\mathbf{GL}_D(V)$

In this section we recall results of [BL02]. We fix a separable field extension $E|k$ in $\mathrm{Lie}(\tilde{\mathbf{G}})(k)$. The group $E^\times$ acts on $\mathrm{Latt}^2_{o_k}(A)$ by conjugation, i.e. there is an $E^\times$-action on $\mathfrak{B}(\tilde{\mathbf{G}}, k)$.

Theorem 3.12 *[BL02, II.1.1] There is a unique CLF-application*

$$j: \ \mathfrak{B}(\tilde{\mathbf{G}}, k)^{E^\times} \to \mathfrak{B}(\tilde{\mathbf{G}}_E, k).$$

The map j is bijective and j^{-1} is the unique $\tilde{\mathbf{G}}_E(k)$-equivariant affine map from $\mathfrak{B}(\tilde{\mathbf{G}}_E, k)$ to $\mathfrak{B}(\tilde{\mathbf{G}}, k)$.

Notation 3.13 We denote
$$j_E := j \text{ and } j^E := j^{-1}.$$

In this part of the thesis we mainly consider j^E.

In [BL02] the authors describe j^E. They define a map between the enlarged buildings and apply the projection to the non-enlarged buildings. The projection from an enlarged to the non-enlarged building is
$$(y, w) \mapsto y.$$

Theorem 3.14 *[BL02, II.3.1, II.4] There is an affine $\tilde{\mathbf{G}}_E(k)$-equivariant CLF-map $\tilde{j}$ from $\mathfrak{B}^1(\tilde{\mathbf{G}}_E, k)$ to $\mathfrak{B}^1(\tilde{\mathbf{G}}, k)$, such that the first component of $\tilde{j}(y, w)$ is $j^E(y)$.*

Proof: The existence of $\tilde{j}$ is stated in lemma [BL02, II.3.1]. The affineness is proven in section [BL02, II.4] for j^E, but the proof actually shows that $\tilde{j}$ is affine. The $\mathbf{G}_E(k)$-equivariance follows from the formula of $\tilde{j}$ given in [BL02, II.3.1]. q.e.d.

Corollary 3.15 The image of $\tilde{j}$ is the set of o_D-lattice functions of V which are o_E-lattice functions.

Proof: If $\Lambda \in \mathrm{Latt}^1_{o_D}(V)$ is in the image of $\tilde{j}$ then $\Lambda + l$ is in the image too for every integer l, because of the $k^\times$-equivariance. The affineness implies that a lattice function is an element of $\mathrm{im}(\tilde{j})$ if and only if the whole class is a subset of $\mathrm{im}(\tilde{j})$. The assertion follows now from
$$\mathrm{im}(j^E) = (\mathrm{Latt}_{o_D}(V))^{E^\times}.$$
q.e.d.

One has a uniqueness result for j^E on the level of non-enlarged buildings.

Theorem 3.16 *[BS09, 10.3] For two points $y \in \mathfrak{B}(\tilde{\mathbf{G}}, k)$ and $x \in \mathfrak{B}(\tilde{\mathbf{G}}_E, k)$ which satisfy*
$$\mathrm{LF}(y, \tilde{\mathbf{G}}, k) \cap \mathrm{End}_{E \otimes_k D}(V) \supseteq \mathrm{LF}(x, \tilde{\mathbf{G}}_E, k)$$
we have $j^E(x) = y$.

In [BS09, 10.3] this theorem was proven for the case $D = k$ and E is generated by one element. The proof did not use the second assumption, and it goes over to $D \neq k$ without changes. The theorem generalises easily to semisimple subalgebras.

We assume now that E is a semisimple k-subalgebra of $\mathrm{End}_D(V)$ and E may not be a field. We put $o_E := \sum_i o_{E_i}$.

As in theorem 3.12 we describe $\mathfrak{B}(\tilde{\mathbf{G}}_E, k)$ as a subset of $\mathfrak{B}(\tilde{\mathbf{G}}, k)$. We can not use the action of $E^\times$ because there are no fixed points in $\mathfrak{B}(\tilde{\mathbf{G}}, k)$ if E is not a field. The set of $o_E^\times$-fixed points is too big. We therefore introduce a new notion of lattice function.

Definition 3.17 An *o_E-o_D-lattice function* of V is an o_D-lattice function of V which splits under (V_i) such that for every i the function
$$t \mapsto \Lambda(t) \cap V_i$$

is an o_{E_i}-lattice function of V_i. We denote the set of o_E-o_D-lattice functions by $\mathrm{Latt}^1_{o_E-o_D}(V)$.

Theorem 3.18 *Under the assumptions and notation of subsection 3.2.1 there is an affine and $\tilde{\mathbf{G}}_E(k)$-equivariant CLF-map*

$$\mathfrak{B}^1(\tilde{\mathbf{G}}_E, k) \to \mathfrak{B}^1(\tilde{\mathbf{G}}, k),$$

whose image in terms of lattice functions is $\mathrm{Latt}^1_{o_E-o_D}(V)$.

Proof: For every index i theorem 3.14 ensures the existence of an affine $\mathrm{GL}_{E_i \otimes_k D}(V_i)$-equivariant CLF-map

$$\tilde{j}_i : \mathfrak{B}^1(\mathbf{GL}_{E_i \otimes_k D}(V_i), E_i) \to \mathfrak{B}^1(\mathbf{GL}_D(V_i), k).$$

Thus the product $\tilde{j} := \prod_i \tilde{j}_i$ is an affine and $\tilde{\mathbf{G}}_E(k)$-equivariant CLF-map. The map

$$\oplus_* : \prod_i \mathfrak{B}^1(\mathbf{GL}_D(V_i), k) \to \mathfrak{B}^1(\tilde{\mathbf{G}}, k)$$

defined by

$$(\Lambda_i)_i \mapsto \oplus_i \Lambda_i$$

is affine and $\prod_i \mathrm{GL}_D(V_i)$-equivariant by remark 2.12 and CLF by proposition 2.21. Thus the map

$$j := \oplus_* \circ \tilde{j}$$

fulfils the asserted properties. The assertion about the image follows from corollary 3.15. q.e.d.

3.4. CLF-maps in the case of $\mathbf{O}_{2,k}^{is}$

We consider a local hermitian k-data which satisfies

$$D = k = k_0, \dim_k V = 2, \rho = \mathrm{id}, \text{ Witt index} = 1, \epsilon = 1. \tag{3.4}$$

Remark 3.19 There is only one Witt decomposition and we have a corresponding k-basis v_1, v_2 such that

$$h(v_1\lambda_1 + v_2\lambda_2, v_1\mu_1 + v_2\mu_2) = \lambda_1\mu_2 + \lambda_2\mu_1.$$

The objects are the followings.

1. $\mathbf{O}_{2,k}^{is}(k) = \{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix} \mid \alpha \in k^\times\}$

2. We have $\sigma(B) = \tilde{B}$, $B \in \mathrm{GL}_2(k)$, where $\tilde{B}$ is obtained from B in permuting the diagonal entries.

3. Maps which are compatible with the Lie algebra filtrations

3. The set $\mathrm{Lie}(\mathbf{O}_{2,k}^{is})(k)$ is the set of diagonal matrices $\mathrm{diag}(a,-a)$ where a runs over k. Any element of $\mathrm{Lie}(\mathbf{O}_{2,k}^{is})(k)$ is therefore separable, e.g. for $a \neq 0$ we have

$$k[\mathrm{diag}(a,-a)] = k[\mathrm{diag}(1,-1)] = k\,\mathrm{diag}(1,0) \times k\,\mathrm{diag}(0,1).$$

4. The group $(\mathbf{O}_{2,k}^{is})^0$ is canonically k-isomorphic to $\mathbf{G_m}$ via the following embedding:

$$g \in \mathbf{G_m}(k) \mapsto incl(g) \in \mathbf{O}_{2,k}^{is}(k)$$

where $incl(g)(v_1) = v_1 g$ and $incl(g)(v_2) = v_2 g^{-1}$.

Remark 3.20 Every element of $\mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k)$ has the same Lie algebra filtration, precisely

$$t \mapsto \{\mathrm{diag}(a,-a)|\ a \in \mathfrak{p}_k^{[t]+}\}.$$

Proposition 3.21 *There is an affine map*

$$j:\ \mathfrak{B}^1((\mathbf{O}_{2,k}^{is})_\beta, k) {\rightarrow} \mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k)$$

which is affine, $(\mathbf{O}_{2,k}^{is})_\beta(k)$-equivariant and compatible with the Lie algebra filtrations.

Proof: In the case of $\beta = 0$ we take for j the identity of $\mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k)$. If β is not zero the following map j defined by

$$(\mathfrak{p}_k^{[t+s]+})_{t\in\mathbb{R}} \mapsto (v_1\mathfrak{p}_k^{[t+s]+} + v_2\mathfrak{p}_k^{[t-s]+})_{t\in\mathbb{R}}$$

is compatible with the Lie algebra filtrations with respect to *incl*, because on both sides there is only one filtration and we have

$$d(incl)(\mathfrak{p}_k^{[t]+}) = \{\mathrm{diag}(a,-a) \mid\ a \in \mathfrak{p}_k^{[t]+}\}.$$

The affineness and the equivariance are obvious. q.e.d.

We consider the identifications of remark 2.65.

Proposition 3.22 *A map*

$$j:\ \mathfrak{B}^1((\mathbf{O}_{2,k}^{is})_\beta, k) {\rightarrow} \mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k)$$

which is $\mathbf{G_m}(k)$-equivariant and affine is a translation of $\mathbb{R}$. The translations of $\mathbb{R}$ are in terms of lattice functions $\mathbf{G_m}(k)$-equivariant.

Proof: The linear part of an affine map j on $\mathbb{R}$ must be the identity if j satisfies

$$j(s+1) = j(s) + 1$$

for all $s \in \mathbb{R}$. Thus such a j must be a translation. q.e.d.

Remark 3.23 The identity is the only $\mathbf{O}^{is}_{2,k}(k)$-equivariant translation of $\mathbb{R}$.

3.5. CLF-maps in the unitary case

We use the notation of subsection 3.2.2.

Convention 3.24 We omit the case where $\mathbf{G}$ is k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$, i.e. the case considered in section 3.4. Therefore $\mathbf{G}$ has no proper enlarged building over k_0.

Convention 3.25 We work with the models of the buildings in terms of lattice functions and square lattice functions and we have thus fixed isomorphisms of the form given in theorem 2.27 and theorem 2.64.

Theorem 3.26 *There is an injective, affine and* $\mathbf{H}(k_0)$*-equivariant CLF-map*

$$j: \ \mathfrak{B}^1(\mathbf{H}, k_0) \rightarrow \mathfrak{B}(\mathbf{G}, k_0)$$

whose image in terms of lattice functions is the set of self-dual o_E*-*o_D*-lattice functions of* V.

We construct the map using the diagram

$$
\begin{array}{ccc}
\mathfrak{B}^1(\mathbf{H}_{un}, k_0) \times \mathfrak{B}^1(\mathbf{H}_{GL}, k_0) & \overset{\phi}{\rightarrow} & \mathfrak{B}^1(\mathbf{G}_{un}, k_0) \times \mathfrak{B}^1(\mathbf{G}_{GL}, k_0) \\
\| & & \psi \downarrow \\
\mathfrak{B}^1(\mathbf{H}, k_0) & \overset{j}{\rightarrow} & \mathfrak{B}(\mathbf{G}, k_0)
\end{array}
$$

We have to construct ϕ and ψ.

Lemma 3.27 There is an injective, affine and $\mathbf{G}_{un}(k_0) \times \mathbf{G}_{GL}(k_0)$-equivariant CLF-map ψ.

Proof: We define ψ to be the map

$$\mathrm{Latt}^1_{h_{un}}(V_{un}) \times \mathrm{Latt}^1_{h_{GL}}(V_{GL}) \rightarrow \mathrm{Latt}^1_h(V)$$

given by

$$(\Lambda_{un}, \Lambda_{GL}) \mapsto \Lambda_{un} \oplus \Lambda_{GL}.$$

The map is welldefined by 2. of remark 2.49 because $h(V_{un}, V_{GL}) = \{0\}$. The affineness, the equivariance and the injectivity are obvious. The CLF-property follows from proposition 2.21 and

$$\mathrm{Skew}(\mathrm{End}_D(V), \sigma_h) \cap (\mathrm{End}_D(V_{un}) \oplus \mathrm{End}_D(V_{GL})) =$$

$$\mathrm{Skew}(\mathrm{End}_D(V_{un}), \sigma_{un}) \oplus \mathrm{Skew}(\mathrm{End}_D(V_{GL}), \sigma_{GL}).$$

q.e.d.

3. Maps which are compatible with the Lie algebra filtrations

Lemma 3.28 There is an injective, affine and $\mathbf{H}_{un}(k_0)$-equivariant CLF-map

$$\phi_{un} : \ \mathfrak{B}^1(\mathbf{H}_{un}, k_0) \rightarrow \mathfrak{B}^1(\mathbf{G}_{un}, k_0).$$

Proof:

1. At first we assume that E_{un} is a field. We use diagram (2.8) and in terms of lattice functions we consider $\mathfrak{B}^1(\mathbf{H}_{un}, k_0)$ and $\mathfrak{B}^1(\mathbf{G}_{un}, k_0)$ as embedded in $\mathfrak{B}(\tilde{\mathbf{H}}_{un}, k)$ and $\mathfrak{B}(\tilde{\mathbf{G}}_{un}, k)$ respectively. We have to prove that the image of $\mathfrak{B}^1(\mathbf{H}_{un}, k_0)$ under $j^{E_{un}}$ of theorem 3.12 is a subset of $\mathfrak{B}^1(\mathbf{G}_{un}, k_0)$. For an element x of $\mathfrak{B}^1(\mathbf{H}_{un}, k_0)$ we have

$$\mathrm{LF}(x, \mathrm{Lie}(\tilde{\mathbf{H}}_{un}), k) = \mathrm{LF}(j^{E_{un}}(x), \mathrm{Lie}(\tilde{\mathbf{G}}_{un}), k) \cap \mathrm{End}_{E_{un} \otimes_k D}(V_{un})$$

by the CLF-property of $j^{E_{un}}$. The left hand side is invariant under σ, and we obtain

$$\mathrm{LF}(x, \mathrm{Lie}(\tilde{\mathbf{H}}_{un}), k) = \mathrm{LF}(j^{E_{un}}(x), \mathrm{Lie}(\tilde{\mathbf{G}}_{un}), k)^\sigma \cap \mathrm{End}_{E_{un} \otimes_k D}(V_{un}). \qquad (3.5)$$

By 2.37[4.] there is a point y of $\mathfrak{B}(\tilde{\mathbf{G}}_{un}, k)$ whose Lie algebra filtration is

$$\mathrm{LF}(j^{E_{un}}(x), \mathrm{Lie}(\tilde{\mathbf{G}}_{un}), k)^\sigma.$$

By theorem 3.16 the equation (3.5) implies $y = j^{E_{un}}(x)$, i.e. the Lie algebra filtration of $j^{E_{un}}(x)$ in $\mathrm{End}_D(V_{un})$ is self-dual and therefore $j^{E_{un}}(x)$ lies in $\mathfrak{B}^1(\mathbf{G}_{un}, k_0)$. We define

$$\phi_{un} := j^{E_{un}}|_{\mathfrak{B}^1(\mathbf{H}_{un}, k_0)}.$$

2. If E_{un} is not necessarily a field we get for every $i \in J_{un}$ a map $\phi_{un,i}$ constructed above. The image of $\phi_{un,i}$ is a subset of $\mathrm{Latt}^1_{h|_{V_i \times V_i}}(V_i)$ and we define ϕ_{un} to be the direct sum of the maps $\phi_{un,i}$. The assertion follows now from 2. of remark 2.49.

3. We now prove the injectivity of ϕ_{un}. If two tupel of self-dual lattice functions (Λ_i) and (Λ'_i) are in the same fiber of ϕ_{un} we obtain

$$j^{E_i}([\Lambda_i]) = j^{E_i}([\Lambda'_i])$$

for all indexes i. The injectivity of j^{E_i} implies $[\Lambda_i] = [\Lambda'_i]$ and the self-duality implies $\Lambda_i = \Lambda'_i$.

q.e.d.

Lemma 3.29 There is an injective, affine $\mathbf{H}_{GL}(k_0)$-equivariant CLF-map

$$\phi_{GL} : \ \mathfrak{B}^1(\mathbf{H}_{GL}, k_0) \rightarrow \mathfrak{B}^1(\mathbf{G}_{GL}, k_0).$$

Proof: In terms of lattice functions we have

$$\mathfrak{B}^1(\mathbf{H}_{GL}, k_0) = \mathfrak{B}^1(\tilde{\mathbf{H}}_+, k)$$

and we take a map
$$\phi_{GL,1} : \mathfrak{B}^1(\tilde{\mathbf{H}}_+, k) \to \mathfrak{B}^1(\tilde{\mathbf{G}}_+, k)$$

constructed as in the proof of theorem 3.18. The map

$$\phi_{GL,2} : \ \mathfrak{B}^1(\tilde{\mathbf{G}}_+, k) \to \mathfrak{B}^1(\mathbf{G}_{GL}, k_0)$$

is obtained by proposition 2.55. The CLF property is an easy calculation knowing that

$$\mathrm{Lie}(\tilde{\mathbf{G}}_+)(k) \hookrightarrow \mathrm{Lie}(\mathbf{G}_{GL})(k_0)$$

is given by

$$a \mapsto a \oplus (-a^{\sigma, V_-}).$$

$\phi_{GL,2}$ is injective, affine and $\tilde{\mathbf{G}}_+(k)$-equivariant. We put

$$\phi = \phi_{GL,2} \circ \phi_{GL,1}.$$

q.e.d.

The combination of all lemmas provides the proof of part one of theorem 3.26.

Lemma 3.30 There is an injective, affine and $\mathbf{H}(k_0)$-equivariant CLF-map

$$j : \ \mathfrak{B}^1(\mathbf{H}, k_0) \to \mathfrak{B}(\mathbf{G}, k_0)$$

Proof: We define
$$j := \psi \circ (\phi_{un} \times \phi_{GL}),$$

where $\phi_{GL} := \phi_{GL,2} \circ \phi_{GL,1}$. q.e.d.

In the proof above many choices have been made. The following lemma finishes the proof of theorem 3.26.

Lemma 3.31 Let j be a map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}(\mathbf{G}, k_0)$ constructed as in the proof of 3.30. The image of j is the set of self-dual o_E-o_D-lattice functions.

Proof: **Case 1:** At first we consider the case where J_{un+} has exactly one element.
Case 1.1: If the index lies in J_{un} the map j is a restriction of j^E whose image is the set of classes of o_E-o_D-lattice functions by 3.12. Let y be an element of $\mathfrak{B}(\mathbf{G}, k_0)$ whose self-dual lattice function is an o_E-o_D-lattice function. By the surjectivity of j^E there is an $x \in \mathfrak{B}^1(\tilde{\mathbf{H}}, k)$ such that $j^E(x) = y$ and therefore

$$\mathrm{LF}(y, \mathrm{End}_D(V)) \cap \mathrm{End}_{E \otimes_k D}(V) = \mathrm{LF}(x, \mathrm{End}_{E \otimes_k D}(V)).$$

The self-duality of $\mathrm{LF}(y, \mathrm{End}_D(V))$ implies the self-duality of the Lie algebra filtration of x, i.e. x has to be an element of $\mathfrak{B}^1(\mathbf{H}, k_0)$.

Case 1.2: If the unique element of J_{un+} is an element of J_+, then the map j is a composition of

$$(\mathrm{Latt}^1_{o_D}(V_i))^{E_i^\times} \overset{\sim}{\longrightarrow} (\mathrm{Latt}^1_h(V)) \cap (\mathrm{Latt}^1_{o_E - o_D}(V))$$

from proposition 2.55 and

$$\tilde{j} : \; \mathfrak{B}^1(\mathbf{H}, k_0) = \mathfrak{B}^1((\tilde{\mathbf{G}}_i)_{E_i}, k) \overset{\sim}{\longrightarrow} \mathfrak{B}^1(\tilde{\mathbf{G}}_i, k)^{E_i^\times}.$$

Thus the image of j is the set of self-dual o_E-o_D-lattice functions of V.

Case 2: In the general case j is the direct sum of maps of the kind of the two cases 1.1 and 1.2 which finishes the proof. q.e.d.

4. Uniqueness results

Notation 4.1 We take the notation and assumptions of subsection 3.2.2 and convention 3.24

In theorem 3.26 we proved the existence of an affine and $\mathbf{H}(k_0)$-equivariant CLF-map

$$\mathfrak{B}^1(\mathbf{H}, k_0) \rightarrow \mathfrak{B}(\mathbf{G}, k_0).$$

In this chapter we observe in which sense the CLF property determines such a map if we forget the affineness or weaken the equivariance. It will force uniqueness if $\mathbf{H}$ has no factor k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$ and J_{GL} is empty. It will force a uniqueness up to translations of $\mathfrak{B}^1(\mathbf{H}, k_0)$ in general. For technical reasons we introduce further notation.

Notation 4.2 For $i \in J_{un+}$ we put

- $h_i := h|_{(V_i + V_{-i}) \times (V_i + V_{-i})}$,

- $\mathbf{G}_i := \mathbf{U}(h_i)$ and $\mathbf{H}_i := (\mathbf{G}_i)_{E_i + E_{-i}}$,

- $\tilde{\mathbf{G}}_i := \mathbf{GL}_D(V_i)$ and $\tilde{\mathbf{H}}_i := (\tilde{\mathbf{G}}_i)_{E_i}$.

To shorten the notation we write

- $\tilde{\mathfrak{g}}$ for $\mathrm{Lie}(\tilde{\mathbf{G}})(k)$,

- $\mathfrak{g}$ for $\mathrm{Lie}(\mathbf{G})(k_0)$,

- $\tilde{\mathfrak{h}}$ for $\mathrm{Lie}(\tilde{\mathbf{H}})(k)$ and

- $\mathfrak{h}$ for $\mathrm{Lie}(\mathbf{H})(k_0)$.

For any index i we denote

- $\mathrm{Lie}(\tilde{\mathbf{G}}_i)(k)$ by $\tilde{\mathfrak{g}}_i$,

- $\mathrm{Lie}(\mathbf{G}_i)(k_0)$ by $\mathfrak{g}_i$,

- $\mathrm{Lie}(\tilde{\mathbf{H}}_i)(k)$ by $\tilde{\mathfrak{h}}_i$ and

- $\mathrm{Lie}(\mathbf{H}_i)(k_0)$ by $\mathfrak{h}_i$.

We have to remark the following correspondence. For $i \in J_+$ we have

$$(\mathbf{H}_i)(k_0) = \{a + (a^{-1})^\sigma \mid a \in \operatorname{Aut}_{E_i \otimes_k D}(V_i)\} \cong \operatorname{Aut}_{E_i \otimes_k D}(V_i) = \tilde{\mathbf{H}}_i(k),$$

$$\mathfrak{h}_i = \{a - (a)^\sigma \mid a \in \operatorname{End}_{E_i \otimes_k D}(V_i)\} \cong \operatorname{End}_{E_i \otimes_k D}(V_i) = \tilde{\mathfrak{h}}_i$$

thus a Lie algebra filtration in $\tilde{\mathfrak{h}}_i$ corresponds to a Lie algebra filtration in $\mathfrak{h}_i$. The Lie algebra filtrations are defined in 2.28, 2.66 and 3.10.

4.1. Factorisation

Lemma 4.3 There is at most one index $i \in J$ such that $\beta_i = 0$ and if such an index exists it has to be in J_{un}.

Proof: Assume that there are two different indexes $i, j \in J$ such that β_i and β_j are zero. We take a polynomial P with coefficients in k such that $1_i = P(\beta)$. We obtain firstly

$$1_i = 1_i 1_i = 1_i P(\beta) = 1_i P(0),$$

i.e.

$$(1 - P(0))1_i = 0,$$

and secondly

$$0 = 1_i 1_j = P(\beta)1_j = P(0)1_j.$$

The element $P(0)$ lies in k and therefore it must be 1 by first and 0 by the second equality which gives a contradiction.

If there is one index with $\beta_i = 0$ then $-\beta_{-i} = \beta_i^\sigma$ is zero too, and by the above argument i equals $-i$ and thus $i \in J_{un}$. q.e.d.

For this section, **let $y \in \mathfrak{B}(\mathbf{G}, k_0)$ be an extension of $x \in \mathfrak{B}^1(\mathbf{H}, k_0)$**, i.e.

$$\operatorname{LF}(x, \mathfrak{h}) = \operatorname{LF}(y, \mathfrak{g}) \cap \mathfrak{h}. \tag{4.1}$$

We want to show that $\operatorname{LF}(y, \mathfrak{g})$ is a direct sum of Lie algebra filtrations of $\mathfrak{g}_i$ where i runs over J. The element x is a vector of elements $x_i \in \mathfrak{B}^1(\mathbf{H}_i, k_0)$ for $i \in J_{un+}$.

Lemma 4.4 The idempotents 1_i are elements of $\operatorname{LF}(y, \tilde{\mathfrak{g}})(0)$.

Proof: **Case 1:** We firstly consider an index $i \in J_+$. $1_i - 1_{-i}$ is an element of $\operatorname{LF}(x_i, \mathfrak{h}_i)(0)$, thus an element of $\operatorname{LF}(y, \mathfrak{g})(0)$ by (4.1) and therefore

$$1_i + 1_{-i} = (1_i - 1_{-i})^2 \in \operatorname{LF}(y, \tilde{\mathfrak{g}})(0).$$

Hence 1_i and 1_{-i} are elements of $\operatorname{LF}(y, \tilde{\mathfrak{g}})(0)$ since 2 is invertible in o_k.

Case 2: We take an index $i \in J_{un}$ and we assume that β_i is not zero. Since $\beta_i \in E_i$ is skewsymmetric we have for all $t \in \mathbb{R}$

$$\beta_i \operatorname{LF}(x_i, \tilde{\mathfrak{h}}_i)(t) = \operatorname{LF}(x_i, \tilde{\mathfrak{h}}_i)(t + \nu(\beta_i))$$

and

$$\beta_i \operatorname{LF}(x_i, \mathfrak{h}_i)(t) = \operatorname{LF}(x_i, \tilde{\mathfrak{h}}_i)(t + \nu(\beta_i)) \cap \operatorname{Sym}(\tilde{\mathfrak{h}}_i, \sigma_i).$$

By the invertibility of 2 in o_k every element of $\operatorname{LF}(x_i, \tilde{\mathfrak{h}}_i)(t)$ is a sum of a skewsymmetric and a symmetric element of $\operatorname{LF}(x_i, \tilde{\mathfrak{h}}_i)(t)$. Thus we obtain

$$\begin{aligned}
\operatorname{LF}(x_i, \tilde{\mathfrak{h}}_i)(0) &= \operatorname{LF}(x_i, \mathfrak{h}_i)(0) + \beta_i \operatorname{LF}(x_i, \mathfrak{h}_i)(-\nu(\beta_i)) \\
&\subseteq \operatorname{LF}(y, \mathfrak{g})(0) + \operatorname{LF}(y, \mathfrak{g})(\nu(\beta_i)) \operatorname{LF}(y, \mathfrak{g})(-\nu(\beta_i)) \\
&\subseteq \operatorname{LF}(y, \tilde{\mathfrak{g}})(0)
\end{aligned}$$

and the ith idempotent 1_i is an element of $\operatorname{LF}(y, \tilde{\mathfrak{g}})(0)$.

Case 3: If there is an index i_0 such that $\beta_{i_0} = 0$, by lemma 4.3 it is unique, and the two cases above imply

$$1_{i_0} = 1 - \sum_{i \neq i_0} 1_i \in \operatorname{LF}(y, \tilde{\mathfrak{g}})(0).$$

q.e.d.

The idea of the proof of case 2 is taken from [BS09, 11.2]..

Corollary 4.5 The o_D-lattice function of y splits under $(V_i)_{i \in J}$ and y is in the image of the injective, affine and $\prod_{i \in J_{un+}} \mathbf{G}_i(k_0)$-equivariant CLF-map

$$\psi_J : \prod_{i \in J_{un+}} \mathfrak{B}^1(\mathbf{G}_i, k_0) \rightarrow \mathfrak{B}(\mathbf{G}, k_0)$$

which is defined by taking the direct sum of the self-dual lattice functions.

The fixed element y is in the image of ψ_J, i.e.

$$y = \psi_J((y_i)_{i \in J_{un+}})$$

for some

$$(y_i) \in \mathfrak{B}^1(\prod_{i \in J_{un+}} \mathbf{G}_i, k_0) = \prod_{i \in J_{un+}} \mathfrak{B}^1(\mathbf{G}_i, k_0).$$

We now prove property (4.1) for the coordinates.

Lemma 4.6 For all $i \in J_{un+}$ we have

$$\operatorname{LF}(x_i, \mathfrak{h}_i) = \mathfrak{h}_i \cap \operatorname{LF}(y_i, \mathfrak{g}_i).$$

Proof: Let $t \in \mathbb{R}$. We have

$$\prod_i \mathrm{LF}(x_i, \mathfrak{h}_i)(t) = \mathrm{LF}(x, \mathfrak{h})(t)$$

$$= \mathrm{LF}(y, \mathfrak{g})(t) \cap \mathfrak{h}$$

$$= (\mathrm{LF}(y, \mathfrak{g})(t) \cap \prod_i \mathfrak{g}_i) \cap \mathfrak{h}$$

$$= (\prod_i \mathrm{LF}(y_i, \mathfrak{g}_i)(t)) \cap (\prod_i \mathfrak{h}_i)$$

$$= \prod_i (\mathrm{LF}(y_i, \mathfrak{g}_i)(t) \cap \mathfrak{h}_i).$$

Thus we have for all indexes $i \in J_{un+}$ and for all $t \in \mathbb{R}$ the property

$$\mathrm{LF}(x_i, \mathfrak{h}_i)(t) = \mathrm{LF}(y_i, \mathfrak{g}_i)(t) \cap \mathfrak{h}_i.$$

q.e.d.

The last two lemmatas lead to a factorization of a CLF-map. More precisely we can prove the following proposition.

Proposition 4.7 *If j is a CLF-map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}(\mathbf{G}, k_0)$ there is a unique map*

$$\tau : \ \mathfrak{B}^1(\mathbf{H}, k_0) {\rightarrow} \mathfrak{B}^1(\prod_{i \in J_{un+}} \mathbf{G}_i, k_0) \ \textit{such that } j = \psi_J \circ \tau.$$

The map τ is

1. *a CLF-map,*

2. *affine if j is affine, and*

3. *$\mathbf{H}(k_0)$-equivariant if j is $\mathbf{H}(k_0)$-equivariant.*

Proof: The value $j(x')$ of a point x' is an extension of x' and lies in the image of ψ_J by corollary 4.5. The injectivity of ψ_J implies the unique existence of τ. In addition to the injectivity the map ψ_J is affine and $\prod_{i \in J_{un+}} \mathbf{G}_i(k_0)$-equivariant which implies 2 and 3. q.e.d.

Remark 4.8 The proposition allows us to reduce proofs to the case where J_{un+} has only one element, i.e. where E is a field or a product of two fields which are switched by σ. The first case corresponds to a non-empty J_{un} and the second case to a non-empty J_+.

4.2. Uniqueness if J_{GL} is empty

Theorem 4.9 *Assume that J_{GL} is empty and that no $\mathbf{H}_i$ is k_0-isomorphic to $\mathbf{O}_{2,k_0}^{is}$. There is exactly one CLF-map j^β from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}(\mathbf{G}, k_0)$. Indeed we have the*

stronger result that $j^\beta(x) = y$ if y is an extension of x.

By remark 4.8 it is enough to prove the theorem for the case where E is a field. We only have to ensure that $\mathbf{O}^{is}_{2,k_0}$ does not occur among the $\mathbf{G}_i$.

Lemma 4.10 Under the assumptions of theorem 4.9 no group $\mathbf{G}_i$ is k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$.

Proof: If $\mathbf{G}_i$ is k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$ then β_i has to be zero by remark 3.19[3.] because E_i is stable under σ, i.e. $\mathbf{H}_i$ equals $\mathbf{G}_i$ and is k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$ which is excluded by the assumption of the theorem. q.e.d.

Theorem 4.9 will be proven by two steps.

Proposition 4.11 *(Compare with [BS09, 11.2]) Theorem 4.9 is true if E is a field and β is not zero.*

Proof: If y is an extension of x we have by the same argument as in case 2 in the proof of lemma 4.4 that

$$\mathrm{LF}(x, \tilde{\mathfrak{h}})(t) \subseteq \mathrm{LF}(y, \tilde{\mathfrak{g}})(t).$$

We now apply theorem 3.16 and obtain

$$y = j^E(x)$$

if we consider x as an element of $\mathfrak{B}(\tilde{\mathbf{H}}, E)$ and y as an element of $\mathfrak{B}(\tilde{\mathbf{G}}, k)$. Thus for every $x \in \mathfrak{B}^1(\mathbf{H}, k_0)$ there is only one extension in $\mathfrak{B}(\mathbf{G}, k_0)$. q.e.d.

Lemma 4.12 The theorem is true if β is zero.

For the proof we need the following operation on square matrices.

Definition 4.13 For a square matrix $B = (b_{i,j}) \in M_r(D)$ the matrix $\tilde{B}$ is defined to be $(b_{r+1-j,r+1-i})_{i,j}$, i.e. $\tilde{B}$ is obtained from B by a reflection on the antidiagonal.

Proof: If σ is of the second type there is a skewsymmetric non-zero element β' in k and we can replace β by β' and apply theorem 4.11. Thus we only need to consider σ to be of the first kind. We fix a point $y \in \mathfrak{B}(\mathbf{G}, k_0)$ and fix an apartment containing y. This apartment is determined by a Witt decomposition and thus determined by a Witt basis (w_i) by corollary 1.20. The self-dual o_D-lattice function Λ corresponding to y is split by this basis and is thus described by its intersections with the lines $w_i D$, i.e. there are real numbers α_i such that

$$\Lambda(t) = \bigoplus_i v_i \mathfrak{p}_D^{[(t-\alpha_i)d]+}.$$

4. Uniqueness results

Thus the square lattice function of Λ is

$$\text{End}(\Lambda)(t) = \bigoplus_{i,j} \mathfrak{p}_D^{[(t+\alpha_j-\alpha_i)d]+} E_{i,j}$$

where $E_{i,j}$ denotes the matrix with a 1 in the intersection of the ith row and the jth column and zeros everywhere else. See for example [BL02, I.4.5].

What we have to show is that $\text{End}(\Lambda)$ is determined by the Lie algebra filtration $\text{LF}(y, \mathfrak{g})$. This is enough since the self-dual square lattice function of a point determines the point uniquely. The Gram matrix $\text{Gram}_{(v_i)}(h)$ of the ϵ-hermitian form h has the form

$$\begin{pmatrix} 0 & M & 0 \\ \epsilon M & 0 & 0 \\ 0 & 0 & N \end{pmatrix}$$

with $M := \text{antidiag}(1, \ldots, 1)$ and a diagonal regular matrix N. The adjoint involution of h

$$B \mapsto B^\sigma = \text{Gram}_{(v_i)}(h)^{-1} (B^\rho)^T \text{Gram}_{(v_i)}(h)$$

on $\text{M}_m(D)$ has under this basis the form

$$\begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{C}_{2,2} & \epsilon\tilde{C}_{1,2} & \epsilon M C_{3,2}^T N \\ \epsilon\tilde{C}_{2,1} & \tilde{C}_{1,1} & M C_{3,1}^T N \\ \epsilon N^{-1} C_{2,3}^T M & N^{-1} C_{1,3}^T M & N^{-1} C_{3,3}^T N \end{pmatrix}.$$

The matrices $B_{1,1}, B_{1,2}, B_{2,1}$ and $B_{2,2}$ are $r \times r$-matrices and $C := B^\rho$ where r is the Witt index of h. By the above calculation we obtain that $E_{i,j}^\sigma$ is $+E_{i,j}$, $-E_{i,j}$ or $\lambda E_{u,l}$ with $(i,j) \neq (u,l)$ for some $\lambda \in D^\times$. From the self-duality of $\text{End}(\Lambda)$ and since 2 is invertible in o_k we get:

$$\text{LF}(y, \mathfrak{g})(t) \cap k(E_{i,j} - E_{i,j}^\sigma) = \mathfrak{p}_k^{[t+\alpha_j-\alpha_i]+}(E_{i,j} - E_{i,j}^\sigma).$$

For the calculation see the lemma below. Thus we can get the exponent $\alpha_j - \alpha_i$ from the knowledge of the Lie algebra filtration if $E_{i,j}$ is not fixed by σ. We now consider two cases.

Case 1: We assume that there is an anisotropic part in the Witt decomposition, i.e. N occurs. The matrix $E_{i,m}$ is fixed by σ if and only if i equals m. Thus from the knowledge of the Lie algebra filtration we know all differences $\alpha_i - \alpha_m$ for all indexes i different from m, and thus by subtractions we know the differences $\alpha_i - \alpha_j$ for all i and j.

Case 2: Now we assume that there is no anisotropic part in the Witt decomposition. If ϵ is -1, no $E_{i,j}$ is fixed and we can deduce the differences $\alpha_i - \alpha_j$ for all i and j and, as a consequence, we only have to consider the case where h is hermitian and $D = k$. Here the matrix $E_{i,j}$ is fixed by σ if and only if $i + j = m + 1$. Thus we can determine all differences $\alpha_i - \alpha_j$ where $i + j \neq m + 1$. If m is at least 4 for an index i there is an index $k \neq i$ with $i + k \neq m + 1$ and we can obtain $\alpha_i - \alpha_{m+1-i}$ if we substract $\alpha_k - \alpha_{m+1-i}$ from $\alpha_i - \alpha_k$. The only subcase left is when m equals 2 and ϵ is 1. Here the group $\mathbf{G}$ is k-isomorphic to $\mathbf{O}_{2,k}^{is}$ which is excluded by the assumption of the theorem. q.e.d.

To complete the proof we need the following lemma.

Lemma 4.14 For all $t \in \mathbb{R}$ we have

$$\mathfrak{p}_D^{[td]+} \cap k = \mathfrak{p}_k^{[t]+}.$$

Proof: For an element x of k we have:

$$x \in \mathfrak{p}_D^{[td]+} \text{ if and only if}$$

$$\nu(x) \geq \frac{[td]+}{d}$$

There are integers l and k such that

$$[td]+ = ld + k \text{ and } 1 \leq k \leq d,$$

and thus $[t]+ = l + 1$ and we get that

$$\nu(x) \geq \frac{[td]+}{d} \text{ if and only if } \nu(x) \geq [t] + .$$

The "only if" follows from $\nu(x) \in \mathbb{Z}$. q.e.d.

The proof of theorem 4.9 follows now from lemmas 4.10 and 4.12 and proposition 4.11.

Corollary 4.15 For an index $i \in J_{un}$ the following statements are equivalent.

1. $\mathbf{H}_i$ is k_0-isomorphic to $\mathbf{O}_{2,k_0}^{is}$.

2.
$$\beta_i = 0, k = k_0 = D, \dim_k V_i = 2, \epsilon = 1$$

   and the Witt index of h_i is 1.

3. $\mathbf{G}_i$ is k_0-isomorphic to $\mathbf{O}_{2,k_0}^{is}$.

4. There are at least two CLF-maps from $\mathfrak{B}^1(\mathbf{H}_i, k_0)$ to $\mathfrak{B}^1(\mathbf{G}_i, k_0)$.

5. There are infinitely many CLF-maps from $\mathfrak{B}^1(\mathbf{H}_i, k_0)$ to $\mathfrak{B}^1(\mathbf{G}_i, k_0)$.

Proof: That 1. follows from 3. is a consequence of lemma 4.10. From 1. follows 5. because we have infinitely many translations of $\mathfrak{B}^1(\mathbf{O}_{2,k_0}^{is}, k_0)$. 5. implies 4.. We did not use that $\mathbf{G}$ is not $\mathbf{O}_{2,k_0}$ for the proofs of lemma 4.12 and theorem 4.11. Thus we obtain from 4. the statements 1., 2. and 3.. 3. follows from 2. obviously. We summarise:

$$3. \Rightarrow 1. \Rightarrow 5. \Rightarrow 4. \Rightarrow 2. \Rightarrow 3..$$

q.e.d.

4.3. The image of a CLF-map

Proposition 4.16 *The image of a CLF-map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}(\mathbf{G}, k_0)$ is a subset of the set of o_E-o_D-lattice functions.*

Proof: By proposition 4.7 we can assume that

$$J_{un+} = \{i\}.$$

Case 1: We assume $i \in J_{un}$. If β is zero we have $E = k$ and therefore the $E^\times$-action is trivial. If β is non-zero there is only one CLF-map by theorem 4.9 and it fullfils the assertion by theorem 3.26.

Case 2: We assume $i \in J_+$. Let $y \in \mathfrak{B}(\mathbf{G}, k_0)$ be an extension of $x \in \mathfrak{B}^1(\mathbf{H}, k_0)$. The lattice function Λ of y splits under (V_i, V_{-i}) by corollary 4.5 and by the self duality we only have to pove that $\Lambda \cap V_i$ is an o_{E_i}-lattice function. The building

$$\mathfrak{B}^1(\mathbf{H}, k_0) = \mathfrak{B}^1(\mathbf{GL}_{E_i \otimes_k D}(V_i), E_i)$$

is identified with the set of lattice functions over a skewfield whose center is E_i. Thus we get

- $a - a^\sigma \in \mathrm{LF}(x, \mathfrak{h})(0) \subseteq \mathrm{LF}(y, \mathfrak{g})(0)$ for all $a \in o_{E_i}^\times$,

- $\pi_{E_i} - \pi_{E_i}^\sigma \in \mathrm{LF}(x, \mathfrak{h})(\frac{1}{e}) \subseteq \mathrm{LF}(y, \mathfrak{g})(\frac{1}{e})$ and

- $\pi_{E_i}^{-1} - (\pi_{E_i}^{-1})^\sigma \in \mathrm{LF}(x, \mathfrak{h})(-\frac{1}{e}) \subseteq \mathrm{LF}(y, \mathfrak{g})(-\frac{1}{e})$

where e is the ramification index of $E_i|k$ and π_{E_i} is a prime element of E_i. We conclude that $1_i\Lambda$ is an o_{E_i}-lattice function. q.e.d.

4.4. Rigidity of Euclidean buildings

Definition 4.17 Let S be a set with affine structure. An *affine functional f* on S is an affine map from S to $\mathbb{R}$, i.e.

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

for all $t \in [0, 1]$ and $x, y \in S$.

We analyse affine functionals on the buildings $\mathfrak{B}(\mathbf{G}, k_0)$ and $\mathfrak{B}^1(\tilde{\mathbf{G}}, k)$. At first we give the general statement.

Proposition 4.18 *Let Ω be a thick Euclidean building and $|\Omega|$ be its geometric realisation, then every affine functional a on $|\Omega|$ is constant.*

For the definition of a Euclidean building and its geometric realisation see [Bro89, VI.3] or chapter 9 in part 2. We use the following properties of a thick Euclidean building in the next proof.

Remark 4.19   1. A building is a chamber complex, especially two arbitrary chambers are connected by a gallery.

2. The thickness, i.e. at every corank 1 face S there are at least three different chambers which have S as a common subface.

3. The geometric realisation of a Euclidean building of rank r has an affine structure and the geometric realisation of an apartment is affine isomorphic to $\mathbb{R}^{r-1}$.

4. For two arbitrary faces there is an apartment containing them.

5. If Σ and Σ' are two apartments containing a chamber C there is an isomorphism of simplicial complexes from Σ to Σ' which fixes the intersection of Σ and Σ'. It induces an affine isomorphism between the geometric realisations.

Proof: (of 4.18) Assume that we are given three vertices P_1, P_2 and P_3 of adjacent chambers C_1, C_2 and C_3, more precisely the three chambers have a common codimension 1 face S and the vertex $P_i \in C_i$ does not lie on S. The line segment $[P_1, P_2]$ meets $[P_1, P_3]$ and $[P_2, P_3]$ in a point $Q \in S$. This is a consequence of 4.19[4., 5.] as follows. We are working in three different apartments simultaneously. If Δ_{ij} denotes an apartment containing C_i and C_j, for different i and j, the affine isomorphism from $|\Delta_{12}|$ to $|\Delta_{13}|$ fixing $|\Delta_{12} \cap \Delta_{13}|$ sends $[P_1, P_2]$ to $[P_1, P_3]$ and thus the unique intersection point in $[P_1, P_2] \cap |\bar{C}_1| \cap |\bar{C}_2|$ lies on $[P_1, P_3]$, and analogously on $[P_3, P_2]$. Without loss of generality assume that $a(Q)$ vanishes. If $a(P_1)$ is negative then $a(P_2)$ and $a(P_3)$ are positive by the affiness of a. Thus $a(Q)$ is positive since it lies on $[P_2, P_3]$. A contradiction. Using galleries we obtain that a is constant on vertices of the same type. An apartment is affinely generated by its vertices of a fixed type. Thus a is constant on every apartment and therefore on $|\Omega|$. q.e.d.

We remind again that $\mathbf{G}$ is not k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$.

Proposition 4.20 *Every affine functional on $\mathfrak{B}(\mathbf{G}, k_0)$ is constant.*

Proof: If $\mathbf{G}$ is totally isotropic then $\mathfrak{B}(\mathbf{G}, k_0)$ is a point and otherwise it is the geometric realisation of a thick Euclidean building. Now we apply proposition 4.18. q.e.d.

Proposition 4.21   *1. A $k^\times$-invariant affine functional a on $\mathfrak{B}^1(\tilde{\mathbf{G}}, k)$ is constant.*

2. Every $k^\times$-invariant affine functional on $\mathfrak{B}^1(\mathbf{O}^{is}_{2,k}, k)$ is constant.

Proof:

1. We can consider a as a map on $\mathfrak{B}(\tilde{\mathbf{G}}, k)$, because the fibers of a are unions of classes of o_D-lattice functions. Now we apply proposition 4.18.

2. It follows from part 1, because

$$\mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k) \cong \mathfrak{B}^1(\mathbf{G}_m, k)$$

by a $k^\times$-equivariant affine bijection.

q.e.d.

4.5. Uniqueness in the general case

In this section we want to generalise theorem 4.9 to the case where there are no restrictions on J. CLF-maps can differ by translations in the following sence.

Definition 4.22 1. Fix a natural number n and a real number s. A *translation* of $\mathfrak{B}^1(\tilde{\mathbf{G}}, k)$ by s is a map
$$t: \ \mathfrak{B}^1(\tilde{\mathbf{G}}, k) \rightarrow \mathfrak{B}^1(\tilde{\mathbf{G}}, k)$$

defined by
$$t(\Lambda) := \Lambda + s$$

in terms of o_D-lattice functions of V. Here $\Lambda + s$ denotes the lattice function

$$r \mapsto \Lambda(r - s).$$

This also defines translations on $\mathfrak{B}^1(\mathbf{O}_{2,k}^{is}, k) \cong \mathfrak{B}^1(\mathbf{G}_m, k)$.

2. We only call the identity of $\mathfrak{B}(\mathbf{G}, k_0)$ a *translation* of $\mathfrak{B}(\mathbf{G}, k_0)$.

3. A *translation* of $\mathfrak{B}^1(\mathbf{H}, k_0)$ is a product of translations t_i of $\mathfrak{B}^1(\mathbf{H}_i, k_0)$ where i runs over J_{un+}.

Remark 4.23 A translation of $\mathfrak{B}^1(\mathbf{H}, k_0)$ is $\mathbf{H}(k_0)$-equivariant if there is no $\mathbf{H}_i$ k_0-isomorphic to $\mathbf{O}_{2,k_0}^{is}$. Otherwise we get $k = k_0$ and such a translation is

$$(\prod_{i \in J_{un+}, \mathbf{H}_i \overset{/k}{\not\cong} \mathbf{O}_{2,k}^{is}} \mathbf{H}_i(k)) \times (\mathbf{O}_{2,k}^{is})^0(k) -$$

and especially $\mathbf{H}^0(k)$-equivariant, but in general not $\mathbf{H}(k)$-equivariant, see 3.23.

Let j be a map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}(\mathbf{G}, k_0)$ constructed as in the proof of theorem 3.26.

Theorem 4.24 *If ϕ is an affine and $Z(\mathbf{H}^0(k_0))$-equivariant CLF-map from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}(\mathbf{G}, k_0)$ then $j^{-1} \circ \phi$ is a translation of $\mathfrak{B}^1(\mathbf{H}, k_0)$. In terms of lattice functions the image of ϕ is the set of self dual o_E-o_D-lattice functions on V and ϕ is $\mathbf{H}^0(k_0)$-equivariant.*

Proof: The image of ϕ is a subset of the image of j which is the set of selfdual o_E-o_D-lattice functions by 4.16 and 3.26, especially $\tau := j^{-1} \circ \phi$ is well-defined. We prove in the lemmas below that τ is a translation. A translation is a bijection and we conclude that ϕ and j have the same image. The $\mathbf{H}^0(k_0)$-equivariance of ϕ follows because j and τ are $\mathbf{H}^0(k_0)$-equivariant. q.e.d.

We work with the notation of the theorem and its proof. The coordinates of τ are denoted by τ_i, $i \in J_{un+}$.

Lemma 4.25 The coordinate τ_i only depends on x_i. For all $i \in J_{un}$ for which $\mathbf{O}^{is}_{2,k_0}$ is not k_0-isomorphic to $\mathbf{H}_i$ we have $\tau_i(x) = x_i$ for all $x \in \mathfrak{B}^1(\mathbf{H}_i, k_0)$.

Proof: We have to look at three cases.
Case 1: For the indexes i in J_{un} for which $\mathbf{H}_i$ is not k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$ we know by theorem 4.9 that $\tau_i(x)$ equals x_i.
Case 2: We assume that we have an index $i \in J_{un}$ such that $\mathbf{H}_i$ is k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$. In this case we have $k = k_0$. A lattice function Λ corresponding to a point of the building $\mathfrak{B}^1(\mathbf{O}_{2,k}, k)$ is identified with a real number, see remark 2.65. If we fix an index $t \in J \setminus \{i\}$ and coordinates x_l for $l \in J \setminus \{t\}$ then the map

$$x_t \mapsto \tau_i(x)$$

is constant by proposition 4.20 or 4.21 and thus τ_i does not depend on x_t.
Case 3: In the case of $i \in J_+$ an analogous argument like in case 2 applies. The affine map we use is the map a_i defined by

$$\Lambda_{\tau_i(x)} = \Lambda_{x_i} + a_i(x).$$

q.e.d.

The last lemma allows us to define a map $\tilde{\tau}_i$ by

$$\tilde{\tau}_i(x_i) := \tau_i(x), \ x \in \mathfrak{B}^1(\mathbf{H}, k_0).$$

Lemma 4.26 The map $\tilde{\tau}_i$ is a translation.

Proof: We firstly consider an index $i \in J_{un}$ such that $\mathbf{H}_i$ is k_0-isomorphic to $\mathbf{O}^{is}_{2,k_0}$. We identify $\mathfrak{B}^1(\mathbf{O}^{is}_{2,k_0}, k_0)$ with $\mathbb{R}$. In this case we have $k = k_0$ and the $\mathrm{SO}_2(k)$-equivariance of τ_i gives

$$\tilde{\tau}_i(x_i + 1) = \tilde{\tau}_i(x_i) + 1.$$

The affineness property implies that $\tilde{\tau}_i$ is a translation. For $i \in J_+$ the map a_i in case 3 of the preceding proof is an affine functional and the $k^\times$-equivariance of τ_i implies the

$k^\times$-invariance of a_i, because one gets in terms of lattice functions

$$\begin{aligned}
\pi_k \Lambda + a_i(\pi_k \Lambda) &= \tilde{\tau}_i(\pi_k \Lambda) \\
&= \pi_k \tilde{\tau}_i(\Lambda) \\
&= \Lambda + a_i(\Lambda) - 1 \\
&= \pi_k \Lambda + a_i(\Lambda)
\end{aligned}$$

. Thus a_i is constant by proposition 4.21. q.e.d.

4.6. Generalisation to the non-separable case

All theorems and propositions of the preceding sections of chapter 3 and 4 work if we forget all the separability assumptions, but we have to explain the definition of the enlarged Bruhat-Tits building of the centraliser. This definition was introduced in [BS09].

Case 1: We firstly summarise changes of subsection 3.2.1. We assume that E is commutative, semisimple and not separable over k. We define

- $\mathfrak{B}(\tilde{\mathbf{G}}_E, k)$ (resp. $\mathfrak{B}^1(\tilde{\mathbf{G}}_E, k)$) to be the product (3.2) (resp. (3.3)),

- $\tilde{\mathbf{G}}_E(k) := \tilde{\mathbf{G}}(k) \cap \tilde{\mathbf{G}}_E$ and

- $\mathrm{Lie}(\tilde{\mathbf{G}}_E)(k) := \mathrm{Z}_{\mathrm{Lie}(\tilde{\mathbf{G}})(k)}(E)$.

Case 2: We now come to the case of a unitary group, i.e. we come to subsection 3.2.2. Let us assume that $k[\beta]$ is semisimple but not separable over k. In this case $\mathbf{H} := \mathbf{G}_\beta$ is well defined but not reductive. We define

- $\mathfrak{B}^1(\mathbf{H}, k_0)$ to be the product

$$\prod_{i \in J_{un}} \mathfrak{B}^1(\mathbf{U}(\sigma|_{\mathrm{End}_{E_i \otimes_k D}(V_i)}), (E_i)_0) \times \prod_{i>0} \mathfrak{B}^1(\mathbf{GL}_{E_i \otimes_k D}(V_i), E_i). \qquad (4.2)$$

- $\mathbf{H}(k_0) := \mathbf{G}(k_0) \cap \mathbf{H}$, $\mathbf{H}^0(k_0) := \mathbf{G}(k_0) \cap \mathbf{H}^0$ and

- $\mathrm{Lie}(\mathbf{H})(k_0) := \mathrm{Z}_{\mathrm{Lie}(\mathbf{G})(k_0)}(\beta)$.

As in 3.9 and 3.10 we define the Lie algebra filtration of a point $x = (x_i)_i$ as the direct sum of the Lie algebra filtrations of the x_i. Also for the non-separable case we have the definition of a CLF-map. A map j between a subset of the (enlarged) building of $\tilde{\mathbf{G}}_E(k)$ and a subset of the (enlarged) building of $\tilde{\mathbf{G}}(k)$ is a *CLF-map* if for every element x of the first and y of the second building with $j(x) = y$ or $j(y) = x$ the equality

$$\mathrm{LF}(y, \tilde{\mathbf{G}}, k)(t) \cap \mathrm{Lie}(\tilde{\mathbf{G}}_E)(k) = \mathrm{LF}(x, \tilde{\mathbf{G}}_E, k)(t)$$

holds for all $t \in \mathbb{R}$. Analogously for the unitary case.

Theorem 4.27 *The theorems 3.26, 4.9 and 4.24 are still valid if one assumes $k[\beta]$ to be semisimple but not necessarily separable over k.*

    Proof: The proofs of the mentioned theorems are valid without changes. q.e.d.

5. Torality

In this chapter we use the notation of subsection 3.2.2 but we skip the assumption that β is separable. We only assume that E is semisimple over k and we apply the conventions and definitions of section 4.6 in the case that β is not separable.

Definition 5.1 A map
$$f : \; \mathfrak{B}^1(G_1, k_0) \rightarrow \mathfrak{B}^1(G_2, k_0)$$
between two enlarged buildings of reductive groups defined over k_0 is called *toral* if for each maximal k_0-split torus S of G_1 there is a maximal k_0-split torus T of G_2 containing S such that f maps the apartment corresponding to S into the apartment corresponding to T. An analogous definition applies to maps between non-enlarged buildings.

Proposition 5.2 *The map j constructed in the proof of theorem 3.26 maps apartments into apartments. Further j is toral if β is separable.*

The proof is divided into two cases. Because of the construction of j as a direct sum of maps it is enough to restrict to the following two cases.

1. case 1: $J_{0+} = J_+ = \{i\}$ and

2. case 2: $J_{0+} = J_0 = \{i\}$.

Proof: [Case 1] We assume that $J_{0+} = J_+ = \{i\}$. By [BL02, 5.1] the map $\phi_{GL,1}$ from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to
$$\mathfrak{B}^1(\mathbf{GL}_D(V_i), k) = \mathfrak{B}^1(\mathrm{Res}_{k|k_0}(\mathbf{GL}_D(V_i)), k_0)$$
mentioned in the proof of lemma 3.29 maps apartments into apartments and further is toral if $E_i|k$ is separable. We prove that the map $\phi_{GL,2}$ from $\mathfrak{B}^1(\mathrm{Res}_{k|k_0}(\mathbf{GL}_D(V_i)), k_0)$ to $\mathfrak{B}^1(\mathbf{G}, k_0)$ is toral. A maximal k-split torus S of $\mathbf{GL}_D(V_i)$ corresponds to a decomposition of V_i in one-dimensional k-subspaces, i.e. there is a decomposition $V_i = \oplus_l V_{i,l}$ such that

$$S(k) = \{g \in \mathrm{GL}_D(V_i)| \; g(V_{i,l}) \subseteq V_{i,l} \text{ for all } l\}.$$

Let $V_{-i,j}$ be the subspace of V_{-i} dual to $V_{i,j}$ and let T be the torus given by the decoomposition
$$V = \oplus_j(V_{i,l} \oplus V_{-i,l}).$$
Under the canonical embedding of $\mathrm{GL}_D(V_i)$ into $\mathrm{GL}_D(V)$

$$g \mapsto g \oplus (g^{\sigma, V_{-i}})^{-1} \tag{5.1}$$

the set $S(k)$ is mapped into $T(k)$ and the image of the apartment of S under

$$\Lambda \in \mathrm{Latt}^1_{o_D}(V_i) \mapsto \Lambda \oplus \Lambda^{\#,V_{-i}} \mathrm{Latt}^1_h(V). \tag{5.2}$$

is a subset of the apartment of T. Let S' and T' be the maximal k_0-split subtori of $\mathrm{Res}_{k|k_0}(S)$ and $\mathrm{Res}_{k|k_0}(T)$ respectively. The set $S'(k_0)$ is mapped into $(T' \cap \mathbf{G})(k_0)$ under (5.1). The image of (5.2) only consists of selfdual lattice functions. Hence $\phi_{GL,2}$ seen as a map from $\mathfrak{B}^1(\mathrm{Res}_{k|k_0}(\mathbf{GL}_D(V_i)), k_0)$ to $\mathfrak{B}^1(\mathbf{G}, k_0)$ is toral. q.e.d.

We make the following definition for the proof of proposition 5.2 in case two.

Definition 5.3 Assume we have given a decomposition

$$V = V'^{+} \oplus V'^{-} \oplus V'^{0}$$

such that V'^{+} and V'^{-} are maximal totally isotropic and $V'^{+} \oplus V'^{-}$ is orthogonal to V'^{0} with respect to h. A maximal k_0-split torus T of $\mathbf{U}(h)$ is *adapted to* (V'^{+}, V'^{-}, V'^{0}) if there is a Witt decomposition (V'^{k}) corresponding to T with anisotropic part V'^{0} such that

$$\oplus_{k>0} V'^{k} = V'^{+} \quad \text{and} \quad \oplus_{k<0} V'^{k} = V'^{-}.$$

An apartment of $\mathfrak{B}^1(\mathbf{G}, k_0)$ is adapted to (V'^{+}, V'^{-}, V'^{0}) if every lattice function in this apartment is split by (V'^{+}, V'^{-}, V'^{0}).

Proof: [Case 2] Here we assume $J_{0+} = J_0 = \{i\}$. Thus we have $E = E_i$. There is a central division algebra Δ over E and a finite dimensional right vector space W such that $\mathrm{End}_{E \otimes_k D}(V)$ is E-algebra isomorphic to $\mathrm{End}_{\Delta}(W)$. We identify the E-algebras $\mathrm{End}_{E \otimes_k D}(V)$ and $\mathrm{End}_{\Delta}(W)$ via a fixed isomorphism and we fix a signed hermitian form h_E which corresponds to the restriction σ_E of σ to the E-algebra $\mathrm{End}_{\Delta}(W)$. Let r be the Witt index of h_E. We fix a decomposition

$$W = (W^{+} \oplus W^{-}) \oplus W^{0} \tag{5.3}$$

such that W^{+} and W^{-} are maximal isotropic subspaces of W contained in the orthogonal complement of W^{0}. Let e_{+}, e_{-} and e_{0} be the projections to the vector spaces W^{+}, W^{-} and W^{0} via the direct sum (5.3). We define

$$V^{+} := e^{+}V, \ V^{-} := e^{-}V \text{ and } V^{0} := e^{0}V.$$

Consider the following diagram.

$$
\begin{array}{ccc}
\mathfrak{B}^1(\mathbf{H}, k_0) & \overset{j}{\to} & \mathfrak{B}^1(\mathbf{G}, k_0) \\
\uparrow & & \uparrow \\
\mathfrak{B}^1(\mathbf{U}((h_E)|_{W^0 \times W^0}), E_0) \times \mathfrak{B}^1(\mathbf{GL}_\Delta(W^+), E) & \overset{\alpha}{\to} & \mathfrak{B}^1(\mathbf{U}(h|_{V^0 \times V^0}), k_0) \times \\
& & \mathfrak{B}^1(\mathbf{GL}_D(V^+), k) \\
\downarrow & & \downarrow \\
\mathfrak{B}(\mathbf{GL}_\Delta(W^+), E) & \to & \mathfrak{B}(\mathbf{GL}_D(V^+), k)
\end{array}
$$

where the rows are induced by j. The lower horizontal arrow fulfils the CLF-property and its image only consists of $E^\times$ fixed points of $\mathfrak{B}(\mathbf{GL}_D(V^+), k)$, both properties inherited from j. Thus the map in the last row is j^E, i.e. the inverse of j_E, because otherwise we could construct a CLF-map from $\mathfrak{B}(\mathbf{GL}_D(V^+), k)^{E^\times}$ to $\mathfrak{B}(\mathbf{GL}_\Delta(W^+), E)$ different from j_E, but such a CLF-map is unique by [BL02, II.1.1.]. Now j^E maps apartments into apartments which implies that j maps apartments adapted to (W^+, W^-, W^0) into apartments adapted to (V^+, V^-, V^0).

We now prove that j is toral if $E|k$ is separable. Let us assume that $E|k$ is separable. This implies that the last row j^E is toral by [BL02, 5.1] which implies the torality of α because the only maximal E_0-split torus of $\mathbf{U}((h_E)|_{W^0 \times W^0})$ is the trivial group. The torality of α implies the torality of j on tori adapted to (W^+, W^-, W^0). Hence j is toral because the triple (W^+, W^-, W^0) was choosen arbitrarily. q.e.d.

6. Summary of the main theorems

In this section we just want to summarise the main results of part 1 in one theorem. See definition 1.22 of a local hermitian datum and see 4.22 for the definition of a translation.

Theorem 6.1 *Let*
$$((A, V, D), \rho, k_0, h, \epsilon, \sigma)$$
be a hermitian datum over a local non-Archimedean field k of residue characteristic different from 2. Let β be an element of $\mathrm{Lie}(\mathbf{U}(h))(k_0)$ such that $E := k[\beta]$ is semisimple over k. We put $\mathbf{G} := \mathbf{U}(h)$ and $\mathbf{H} := \mathbf{G}_\beta$.

1. *There is an injective, affine and $\mathbf{H}(k_0)$-equivariant CLF-map*

$$j : \mathfrak{B}^1(\mathbf{H}, k_0) \rightarrow \mathfrak{B}^1(\mathbf{G}, k_0)$$

   such that:

   a) j maps apartments into apartments and is toral if β is separable,

   b) in terms of lattice functions the image of j is the set of selfdual o_E-o_D-lattice functions,

2. *If j and j' are two affine, $\mathbf{Z}(\mathbf{H}^0(k_0))$-equivariant CLF-maps then there is a translation τ of $\mathfrak{B}^1(\mathbf{H}, k_0)$ such that*
$$j = j' \circ \tau.$$

   Both maps are $\mathbf{H}^0(k_0)$-equivariant and their image is the set of selfdual o_E-o_D-lattice functions.

3. *The k-algebra E is a product of fields E_i. Assume further that every E_i is invariant under σ (i.e. $J = J_{un}$). Let 1_i be the one element of E_i, $V_i := 1_i V$ and let $\mathbf{H}_i$ be the centraliser of $1_i\beta$ in $\mathbf{U}(h|_{V_i \times V_i})$. If no $\mathbf{H}_i$ is k_0-isomorphic to $\mathbf{O}_{2,k_0}^{is}$ then there is exactly one CLF-map j from $\mathfrak{B}^1(\mathbf{H}, k_0)$ to $\mathfrak{B}^1(\mathbf{G}, k_0)$. This map is denoted by j^β.*

This theorem follows from the theorems 3.26, 4.9, 4.24 and 4.27 and the propositions 3.21 , 3.22, 5.2.

Part II.

Embedding types and their geometry

7. Introduction and notation

7.1. First remark

This part answers a question that naturally arises from the papers of M.Grabitz and P. Broussous (see [BG00]) and P. Broussous and B. Lemaire (see [BL02]). For an Azumaya-Algebra A over a non-archimedean local field F, M. Grabitz and P. Broussous have introduced embedding invariants for field embeddings, that is for pairs $(E, \mathfrak{a})$, where E is a field extension of F in A and $\mathfrak{a}$ is a hereditary order which is normalised by $E^\times$. On the other hand if we take such a field extension E and define B to be the centraliser of E in A, then $G := A^\times$ and $G_E := B^\times$ are sets of rational points of reductive groups $\mathbf{G}$ and $\mathbf{H}$ defined over F and E respectively. P. Broussous and B. Lemaire have defined a map $j_E : \mathfrak{B}(\mathbf{G}, F)^{E^\times} \to \mathfrak{B}(\mathbf{H}, E)$, i.e. between the Bruhat-Tits buildings of $\mathbf{G}$ over F and $\mathbf{H}$ over E, see section 2.2 and theorem 10.2. The task Prof. Zink has given to me was to relate the embedding invariants to the behavior of the map j_E with respect to the simplicial structures of $\mathfrak{B}(\mathbf{G}, F)$ and $\mathfrak{B}(\mathbf{H}, E)$.

7.2. Notation

1. The letter ν denotes the valuation on F with $\nu(\pi_F) = 1$.

2. We assume D to be a finite dimensional central division algebra over F of index d.

3. We fix an m dimensional right D vector space V, $m \in \mathbb{N}$, and put $A := \mathrm{End}_D(V)$. In particular V is a left $A \otimes_F D^{op}$-module.

4. The letter L denotes a maximal unramified field extension of F in D and we assume that π_D is a uniformizer of D which normalizes L, i.e. the map

$$x \mapsto \sigma(x) := \pi_D x \pi_D^{-1}, \ x \in D,$$

   generates $\mathrm{Gal}(L|F)$.

5. For a positive integer $f|d$ we denote by L_f the subfield of degree f over F in L.

6. In this part all $o_{F'}$-lattice functions on a vector space over a field F' have period 1, i.e. we have
$$\pi_{F'}\Lambda(x) = \Lambda(x+1).$$
   This is different to part 1.

8. Preliminaries

8.1. Vectors and Matrices up to cyclic permutation

Remark 8.1 All invariants which are considered in this part are vectors or matrices modulo cyclic permutation.

Definition 8.2 Let s be a positive integer and R be an arbitrary non-empty set. Two vectors $w, w' \in R^{1 \times s}$ are said to be *equivalent* if w' can be obtained from w by cyclic permutation of the entries of w, i.e.

$$w' = (w_k, \ldots, w_s, w_1, \ldots, w_{k-1}) \text{ for a } k \in \mathbb{N}_s.$$

The equivalence class is denoted by $\langle w \rangle$.

Vectors: We denote by $\mathrm{Row}(s, t)$ the set of all vectors $w \in \mathbb{N}_0^s$ whose sum of entries is t, where s and t are natural numbers, i.e.

$$\sum_{i=1}^{s} w_i = t.$$

One can represent the class $\langle w \rangle$ of a vector $w \in \mathrm{Row}(s, t)$ by pairs

$$\mathrm{pairs}(\langle w \rangle) := \langle (w_{i_0}, i_1 - i_0), (w_{i_1}, i_2 - i_1), \ldots, (w_{i_k}, i_0 + s - i_k) \rangle,$$

where $(w_{i_j})_{0 \leq j \leq k}$ is the subsequence of the non-zero coordinates. Given a vector w with

$$\mathrm{pairs}(\langle w \rangle) = \langle (a_0, b_0), \ldots, (a_k, b_k) \rangle$$

we define the *complement of* $\langle w \rangle$, denoted by $\langle w \rangle^c$ to be the class $\langle w' \rangle$, such that

$$\mathrm{pairs}(\langle w' \rangle) = \langle (b_0, a_1), (b_1, a_2), (b_2, a_3), \ldots, (b_k, a_0) \rangle.$$

This is a bijection

$$(\ )^c : \mathrm{Row}(s, t)/\sim \ \to \mathrm{Row}(t, s)/\sim .$$

Matrices: For $r, s, t \in \mathbb{N}$, $\mathrm{M}_{r,s}(t)$ denotes the set of $r \times s$-matrices with non-negative integer entries, such that

- in every column there is an entry greater than zero, and

• the sum of all entries is t.

For a matrix $M = (m_{i,j}) \in \mathrm{M}_{r,s}(t)$, we define the vector $\mathrm{row}(M) \in \mathrm{Row}(rs, t)$ to be

$$(m_{1,1}, m_{1,2}, \ldots, m_{1,s}, m_{2,1}, \ldots, m_{2,s}, \ldots, m_{r,s}).$$

Two matrices $M, N \in \mathrm{M}_{r,s}(t)$ are said to be *equivalent* if $\mathrm{row}(M)$ and $\mathrm{row}(N)$ are. The equivalence class is denoted by $\langle M \rangle$.

Example 8.3

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix} \backsim \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \end{pmatrix}$$

8.2. Hereditary orders and lattice chains

In the next section we need the concept of hereditary orders and lattice chains. As references we recommand [Rei03] for hereditary orders and [BL02] for lattice chains. We use definition 2.3 of a full o_D-lattice. We omit the word full.

Definition 8.4 A unital subring $\mathfrak{a}$ of A, is called an o_F-*order of* A if $\mathfrak{a}$ is an o_F-lattice of A. We call an o_F-order $\mathfrak{a}$ *hereditary* if the Jacobson radical $\mathrm{rad}(\mathfrak{a})$ is a projective right-$\mathfrak{a}$ module. The set of all hereditary orders is denoted by $\mathrm{Her}(A)$. For $\mathfrak{a} \in \mathrm{Her}(A)$ we denote by $\mathrm{lattices}(\mathfrak{a})$ the set of all o_D-lattices Γ of V such that $a\Gamma \subseteq \Gamma$ for all $a \in \mathfrak{a}$.

Definition 8.5   1. Let R be a non-empty set, and take $r \in \mathbb{N}$. Given non-empty subsets $R_{i,j}$ of R, $(i,j) \in \mathbb{N}_r^2$, and natural numbers $n_1, \ldots, n_r$, we denote by $(R_{i,j})^{n_1,\ldots,n_r}$ the set of all block matrices in $\mathrm{M}_{\sum_{i=1}^r n_i}(R)$, such that for all (i,j) the (i,j)-block lies in $\mathrm{M}_{n_i,n_j}(R_{i,j})$.

2. Given $r \in \mathbb{N}$, $\bar{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we get a hereditary order

$$\mathfrak{a}^{\bar{n}} := (R_{i,j})^{n_1,\ldots,n_r}, \text{ where}$$

$$R_{i,j} := \begin{cases} o_D, & \text{if } j \leq i \\ \mathfrak{p}_D, & \text{if } i < j \end{cases}.$$

3. A hereditary order of $\mathrm{M}_m(D)$ of this form is called *in standard form*. The class $\langle \bar{n} \rangle$ is called the *invariant* and r the *(simplicial) rank* of $\mathfrak{a}^{\bar{n}}$.

If we say that sets are conjugate to each other, we mean conjugate by an element of $A^{\times}$. The proof of the next theorem is given in [Rei03].

Theorem 8.6 *We fix a D-basis of V and identify A with $\mathrm{M}_m(D)$.*

1. *Two hereditary orders in standard form of A are conjugate to each other if and only if they have the same invariant.*

2. Every $\mathfrak{a} \in \operatorname{Her}(A)$ is conjugate to a hereditary order in standard form.

By this theorem the notion of *invariant* and *rank* carries over to every element of $\operatorname{Her}(A)$ and they do not depend on the choice of the basis.

Definition 8.7 A sequence $(\Gamma_i)_{i \in \mathbb{Z}}$ of lattices of V is called an o_D-*lattice chain in V* if

1. for all integers i, we have $\Gamma_{i+1} \subsetneqq \Gamma_i$, and

2. there exists a natural number r such that for all integers i we have $\Gamma_i \pi_D = \Gamma_{i+r}$.

We call r the *rank of the lattice chain.* For a lattice chain Γ we put

$$\operatorname{lattices}(\Gamma) := \{\Gamma_i \mid i \in \mathbb{Z}\}.$$

Two lattice chains Γ, Γ' are called *equivalent* if lattices(Γ) and lattices(Γ') are equal. We write $[\Gamma]$ for the equivalence class. We define an order by $[\Gamma] \leq [\Gamma']$ if lattices(Γ) is a subset of lattices(Γ'). The set of all lattice chains in V is denoted by $\operatorname{LC}_{o_D}(V)$.

Remark 8.8 For every lattice chain Γ in V, the set

$$\mathfrak{a}_\Gamma := \{a \in A \mid \forall i \in \mathbb{Z} : \ a\Gamma_i \subseteq \Gamma_i\}$$

is a hereditary order of A.

Theorem 8.9 *[BF83, (1.2.8)]* $[\Gamma] \mapsto \mathfrak{a}_\Gamma$ *defines a bijection between the set of equivalence classes of lattice chains in V and the set of hereditary orders of A. We have:*

$$[\Gamma] \leq [\Gamma'] \iff \mathfrak{a}_\Gamma \supseteq \mathfrak{a}_{\Gamma'}$$

In this part we need the definition of $\operatorname{Latt}_{o_D}(V)$ given in 2.17 of part 1. We put $\mathfrak{a}_\Lambda := \operatorname{End}(\Lambda)$ to emphasize that we mainly are interested in a filtration "around" a hereditary order than a lattice function of A, see 2.1.2. We also put

$$\operatorname{rank}([\Lambda]) := \operatorname{rank}(\Lambda),$$

see 2.4, and

$$\operatorname{lattices}(\Lambda) := \operatorname{im}(\Lambda)$$

for

$$\Lambda \in \operatorname{Latt}^1_{o_D}(V).$$

8.3. Embedding types

For a field extension $E|F$ we denote by $E_D|F$ the maximal field extension in $E|F$, which is F-algebra isomorphic to a subfield of L. Its degree is the greatest common divsor of d and the residue degree of $E|F$.

Definition 8.10 An *embedding* is a pair $(E, \mathfrak{a})$ satisfying

1. E is a field extension of F in A,

2. $\mathfrak{a}$ is a hereditary order of A, normalised by $E^\times$.

Two embeddings $(E, \mathfrak{a})$ and $(E', \mathfrak{a}')$ are said to be *equivalent* if there is an element $g \in A^\times$, such that $gE_D g^{-1} = E'_D$ and $g\mathfrak{a}g^{-1} = \mathfrak{a}'$.

Remark 8.11 In each equivalence class of embeddings there is a pair such that the field can be embedded in L.

The definition of $\mathrm{M}_{r,s}(t)$ is in the previous section. Until the end of this section we fix a D-basis of V and identify A with $\mathrm{M}_m(D)$.

Definition 8.12 Let $f|d$ and $r \le m$. A matrix with f rows and r columns is called an *embedding datum* if it belongs to $\mathrm{M}_{f,r}(m)$. Given an embedding datum λ, we define the pearl embedding as follows. The *pearl embedding* of λ (with respect to the fixed D-basis of V) is the embedding $(E, \mathfrak{a})$, with the following conditions:

1. $[E : F] = f$,

2. E is the image of the monomorphism

$$x \in L_f \mapsto \mathrm{diag}(M_1(x), M_2(x), \ldots, M_r(x)) \in \mathrm{M}_m(D)$$

   where
$$M_j(x) = \mathrm{diag}(\sigma^0(x)\mathbb{1}_{\lambda_{1,j}}, \sigma^1(x)\mathbb{1}_{\lambda_{2,j}}, \ldots, \sigma^{f-1}(x)\mathbb{1}_{\lambda_{f,j}})$$

3. $\mathfrak{a}$ is a hereditary order in standard form according to the partition $m = n_1 + \ldots + n_r$ where $n_j := \sum_{i=1}^{f} \lambda_{i,j}$.

Theorem 8.13 *[BG00, 2.3.3 and 2.3.10]*

1. *Two pearl embeddings are equivalent if and only if the embedding data are equivalent.*

2. *In any class of embeddings lies a pearl embedding.*

Definition 8.14 Let $(E, \mathfrak{a})$ be an embedding. By the theorem it is equivalent to a pearl-embedding. The class of the corresponding matrix $(\lambda_{i,j})_{i,j}$ is called the *embedding type* of $(E, \mathfrak{a})$. This definition does not depend on the choice of the basis by the theorem of Skolem-Noether.

9. The simplicial structure of $\mathfrak{B}(\mathbf{GL}_D(V), F)$

In section 2.2 we gave the definition of $\mathfrak{B}^1(\mathbf{GL}_D(V), F)$, i.e. of the Bruhat-Tits building of $\mathbf{GL}_D(V)$ over F. In this part of the thesis we are interested in its simplicial structure.

9.1. Definitions

Here we give the basic definitions in order to be able to state precisely the description of the Euclidean building of $\mathbf{GL}_D(V)$ over F with lattice chains. Basic definitions of the notions of simplicial complex and chamber complex are given in [Bro89, Ch. I App.]. For the definition of a Coxeter complex see [Bro89, Ch. III].

Definition 9.1 A *building* is a triple $(\Omega, \mathcal{A}, \leq)$, such that $(\Omega, \leq)$ is a simplicial complex and $\mathcal{A}$ is a set of subcomplexes of $(\Omega, \leq)$ which cover Ω, i.e.

$$\bigcup \mathcal{A} = \Omega,$$

(The elements of $\mathcal{A}$ are called *apartments.*) statifying the following *building axioms*:

- **B0** Every element of $\mathcal{A}$ is a Coxeter complex.

- **B1** For *faces* (also called *simplicies*), i.e. elements, S_1 and S_2 of Ω there is an apartment Σ containing them.

- **B2** If Σ and Σ' are two arpartments containing S_1 and S_2 then there is a poset isomorphism from Σ to Σ' which fixes $\bar{S}_1$ and $\bar{S}_2$ where $\bar{S}$ for a face S is defined to be the set of all faces $T \leq S$.

The minimal faces are the *vertices* and the rank of a face S is the number of vertices in $\bar{S}$. The maximal faces are the *chambers*. Faces of rank two are *edges*. A building is said to be *thick* if every codimension 1 face is attached to at least three chambers.

Remark 9.2 A building in this part of the thesis consist either only of one element or is thick.

A *Euclidean Coxeter complex* is a Coxeter complex $(\Sigma, \leq)$ which is poset-isomorphic to a simplicial complex $\Sigma(W, V)$ defined by an essential irreducible infinite affine reflection group (W, V). For a face S of a simplicial complex $(\Omega, \leq)$ the set of all formal sums

$$\Sigma_{v \leq S, rk(v)=1} \lambda_v v$$

with positive real coefficients such that $\Sigma_{v \leq S, rk(v)=1} \lambda_v = 1$ is denoted by $|S|$. The set

$$|\Omega| := \bigcup_{S \in \Omega} |S|$$

is called the geometric realisation of Ω. A *morphism* of simplicial complexes from $(\Omega, \leq)$ to $(\Omega', \leq')$ is a map $f : (\Omega, \leq) \to (\Omega', \leq')$, such that for every face $S \in \Omega$ the restriction $f : \bar{S} \to f(S)$ is a poset isomorphism. In [Bro89] the notion of non-degenerate simplicial map is used instead of morphism. A morphism f induces a map $|f|$ between the geometric realisations, by

$$|f|(\sum_v \lambda_v v) := \sum_v \lambda_v f(v).$$

Definition 9.3 Given two buildings $(\Omega, \mathcal{A}, \leq)$ and $(\Omega', \mathcal{A}', \leq')$ a *morphism* from the first to the latter is a morphism of simplicial complexes such that the image of an apartment of $\mathcal{A}$ is contained in an apartment of $\mathcal{A}'$.

As described in [Bro89] VI.3 there is a canonical way to define a metric, up to a scalar, on the geometric realisation of a Euclidean building by pulling back the metric from an affine reflection group to the apartment and this defines a canonical affine structure on the geometric realisation of the building. The map $|\phi|$ between the geometric realisations of two Euclidean buildings induced by an isomorphism ϕ is affine.

9.2. The description with lattice chains

Let Ω be the simplicial structure of $\mathfrak{B}(\mathbf{GL}_D(V), F)$. We denote

$$\mathcal{I} := |\Omega| = \mathfrak{B}(\mathbf{GL}_D(V), F).$$

By theorem 2.27 there is a unique affine and $A^\times$-equivariant bijection from $\mathcal{I}$ to $\mathrm{Latt}_{o_D}(V)$. We describe the Euclidean building Ω of $A^\times$ in terms of lattice chains and hereditary orders as it is done in [BL02, I.3].

Proposition 9.4   *1. The posets* $(\mathrm{LC}_{o_D}(V), \leq)$ *and* $(\mathrm{Her}(A), \supseteq)$ *are simplicial complexes of rank m. They are isomorphic via* $\Psi([\Gamma]) := \mathfrak{a}_\Gamma$ *as simplicial complexes.*

   2. A hereditary order is a vertex (resp. a chamber) if and only if its rank is 1 (resp. m).

Definition 9.5 A *frame of V* is a set of lines $v_1 D, \ldots, v_m D$, where $v_i, i \in \mathbb{N}_m$, is a D-basis of V. If $\mathfrak{R}$ is a frame we say that a lattice Γ *is split by* $\mathfrak{R}$ if

$$\Gamma = \bigoplus_{W \in \mathfrak{R}} (\Gamma \cap W).$$

A lattice chain Γ, lattice function Λ, hereditary order $\mathfrak{a}$ is *split by* $\mathfrak{R}$ if every element of lattices(Γ), lattices(Λ), lattices($\mathfrak{a}$) resp. is split by $\mathfrak{R}$. An equivalence class is *split by* $\mathfrak{R}$ if every element of the equivalence class is split by $\mathfrak{R}$. The set of these classes split by $\mathfrak{R}$ is called the *apartment corresponding to* $\mathfrak{R}$ and is denoted by $\mathrm{LC}_{o_D}(V)_{\mathfrak{R}}$, $\mathrm{Her}(A)_{\mathfrak{R}}$, $\mathrm{Latt}_{o_D}(V)_{\mathfrak{R}}$ resp.. For the set of these apartments we write

$$\mathfrak{A}(\mathrm{LC}_{o_D}(V)), \ \mathfrak{A}(\mathrm{Her}(A)) \ \& \ \mathfrak{A}(\mathrm{Latt}_{o_D}(V)).$$

Definition 9.6 The left action of $A^{\times}$ on the set of o_D-lattices of V, i.e.

$$g.\Gamma := \{g\gamma| \ \gamma \in \Gamma\},$$

defines an $A^{\times}$-action on $\mathrm{LC}_{o_D}(V)$, $\mathrm{Latt}_{o_D}(V)$ and $\mathrm{Her}(A)$.

Proposition 9.7   *1. The two triples*

$$(\mathrm{LC}_{o_D}(V), \mathfrak{A}(\mathrm{LC}_{o_D}(V)), \leq) \ \& \ (\mathrm{Her}(A), \mathfrak{A}(\mathrm{Her}(A)), \supseteq)$$

are isomorphic Euclidean buildings via Ψ.

2. Ψ *is* $A^{\times}$-*equivariant.*

3. For every frame $\mathfrak{R}$ *the image of* $\mathrm{LC}_{o_D}(V)_{\mathfrak{R}}$ *under* Ψ *is* $\mathrm{Her}(A)_{\mathfrak{R}}$.

For steps and calculations for the proof see for example [Rei03].

Remark 9.8 Every $\mathfrak{a} \in \mathrm{Her}(A)$ has a rank as a face in the chamber complex $(\mathrm{Her}(A), \supseteq)$, and this coinsides with the simplicial rank, but we never mean the o_F-rank of $\mathfrak{a}$.

From now on we need the affine structure on $\mathrm{Latt}_{o_D}(V)$, see definition 2.17. Now the next proposition explains why one can replace Ω by the building of classes of lattice functions. The geometric realisation of $\mathrm{LC}_{o_D}(V)$ can be identified with $\mathrm{Latt}_{o_D}(V)$ in the following way. We put

$$[x]+ := \inf\{z \in \mathbb{Z}| \ x \leq z\}, \ x \in \mathbb{R},$$

and we define a bijective map

$$\tau : \ |\mathrm{LC}_{o_D}(V)| \rightarrow \mathrm{Latt}_{o_D}(V)$$

as follows. A convex barycenter

$$\sum \beta_i[\Gamma^i], \ \beta_i \geq 0 \ \text{and} \ \sum_i \beta_i = 1,$$

with vertices $[\Gamma^i]$ of a chamber of $\mathrm{LC}_{o_D}(V)$ is mapped to $\sum_i \beta_i[\Lambda^i]$, where $\Lambda^i(t) := \Gamma_0^i \mathfrak{p}_D^{[td]+}$.

Remark 9.9 The definition of τ and proposition 2.29 imply that $\mathrm{Latt}_{o_D}(V)$ inherits the same simplicial structure from Ω and from $\mathrm{LC}_{o_D}(V)$.

Proposition 9.10 ([BL02] sec. I.3) *The composition of the bijection from* $\mathrm{Latt}_{o_D}(V)$ *to* $\mathcal{I}$ *with* τ *induces an* $A^\times$*-equivariant isomorphism from*

$$(\mathrm{LC}_{o_D}(V), \mathfrak{A}(\mathrm{LC}_{o_D}(V)), \leq)$$

to the building Ω.

Notation 9.11 By the propositions above we can identify Ω with

$$(\mathrm{Her}(A), \mathfrak{A}(\mathrm{Her}(A)), \supseteq).$$

Definition 9.12 We call Ω the *Euclidean building of* $A^\times$.

10. The map j_E

Notation 10.1 For this section let $E|F$ be a field extension in A and we set B to be the centraliser of E in A, i.e.

$$B := Z_A(E) := \{a \in A|\ ab = ba\ \forall b \in B\}.$$

We denote the Euclidean building of $B^\times$ by Ω_E and its geometric realisation by $\mathcal{I}_E$.

The next results are taken from [BL02]. We restate the following theorem in the notation of this part.

Theorem 10.2 *[BL02, Thm. II.1.1.] There exists a unique application $j_E : \mathcal{I}^{E^\times} \to \mathcal{I}_E$ such that for any $x \in \mathcal{I}^{E^\times}$ and $t \in \mathbb{R}$ we have $\mathfrak{a}_{j_E(x)}(t) = B \cap \mathfrak{a}_x(e(E|F)t)$. The map j_E satisfies the following properties:*

1. it is bijecive,

2. it is a $B^\times$-equivariant and

3. it is affine.

Moreover its inverse j_E^{-1} is the only map $\mathcal{I}_E \to \mathcal{I}$ such that 2. and 3. hold.

We briefly give Broussous and Lemaire's description of j_E in terms of lattice functions but only in the case, where $E|F$ is isomorphic to a subextension $L_f|F$ of $L|F$. Then $E \otimes_F L \cong \bigoplus_{k=0}^{f-1} L$ coming from the decomposition $1 = \sum_{k=0}^{f-1} 1_k$ labeled such that the $\mathrm{Gal}(L|F)$-action on the second factor gives $\sigma(1_k) = 1_{k-1}$ for $k \geq 1$ and $\sigma(1_0) = 1_{f-1}$. Applying it on the $E \otimes_F L$-module V, we get $V = \bigoplus_k V_k$, $V_k := 1_k V$.

Remark 10.3   1. $B \cong \mathrm{End}_{\Delta_E}(V_0)$ and

   2. $B \cong \mathrm{M}_m(\Delta_E)$
where $\Delta_E := Z_D(L_f)$.

Theorem 10.4 *[BL02, II 3.1.] In terms of lattice functions j_E has the form $j_E^{-1}([\Theta]) = [\Lambda]$, with*

$$\Lambda(s) := \bigoplus_{k=0}^{f-1} \Theta(s - \frac{k}{d})\pi_D^k,\ s \in \mathbb{R}$$

where Θ is an o_Δ-lattice function on V_0.

11. Embedding types through barycentric coordinates

In this chapter we keep the notation from chapter 10. We repeat that $E_D|F$ denotes the biggest field extension of $E|F$ which can be embedded in $L|F$. The centraliser of E_D in A is denoted by B_D. We need a notion of orientation on Ω_{E_D} to order the barycentric coordinates of a point in $\mathcal{I}_{E_D}$.

Definition 11.1 An edge of Ω with vertices e and e' is *oriented towards* e', if there are lattices $\Gamma \in \text{lattices}(e)$ and $\Gamma' \in \text{lattices}(e')$, such that $\Gamma \supseteq \Gamma'$ with the quotient having κ_D-dimension 1, i.e. κ_F-dimension d. We write $e{\rightarrow}e'$. If x is a point in $\mathcal{I}$ then there is a chamber $C \in \Omega$ such that x lies in the closure of $|C|$, i.e. in

$$\bigcup_{S \leq C} |S|.$$

The vertices of C can be given in the way

$$e_1{\rightarrow}e_2{\rightarrow}\ldots{\rightarrow}e_m{\rightarrow}e_1.$$

If (μ_i) are the barycentric coordinates of x with respect to (e_i), i.e.

$$x = \sum_i \mu_i e_i,$$

then the class $\langle \mu \rangle$ is called the *local type of x*.

This definition applies for $\mathcal{I}_{E_D}$ as well. The skewfield is then $\mathrm{Z}_D(E_D)$ instead of D and one has to substitute d by $\frac{d}{[E_D:F]}$.

Proposition 11.2 *The notion of local type does not depend on the choice of the chamber C and the starting vertex e_1.*

For the definition of $\langle \rangle^c$ see section 8.1.

Theorem 11.3 *Let $(E, \mathfrak{a})$ be an embedding of A with embedding type $\langle \lambda \rangle$ and suppose $\mathfrak{a}$ to have rank r. If $M_{\mathfrak{a}}$ denotes the barycenter of $\mathfrak{a}$ in $\mathcal{I}$ and $\langle \mu \rangle$ the local type of $j_{E_D}(M_{\mathfrak{a}})$, then the following holds.*

1. $r[E_D : F]\mu \in \mathbb{N}_0^m$, and

2. $\langle \text{row}(\lambda) \rangle = \langle [E_D : F]r\mu \rangle^c$.

11. Embedding types through barycentric coordinates

Remark 11.4 With theorem 11.3 we can calculate the embedding type from the local type. For example take $r = 2$, $f = 6$, $m = 7$ and assume that $j_{E_D}(M_{\mathfrak{a}})$ is

$$\frac{3}{12}b_0 + \frac{2}{12}b_1 + \frac{1}{12}b_2 + \frac{0}{12}b_3 + \frac{0}{12}b_4 + \frac{4}{12}b_5 + \frac{2}{12}b_6.$$

and thus
$$\langle 12\mu \rangle = \langle 3, 2, 1, 0, 0, 4, 2 \rangle \equiv \langle (3,1), (2,1), (1,3), (4,1), (2,1) \rangle.$$

From the complement

$$\langle 12\mu \rangle^c \equiv \langle (1,2), (1,1), (3,4), (1,2), (1,3) \rangle \equiv \langle 1, 0, 1, 3, 0, 0, 0, 1, 0, 1, 0, 0 \rangle$$

applying theorem 11.3 we can deduce the embedding type of $(E, \mathfrak{a})$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the proof we can restrict to the case where $E = E_D$ and thus $B = B_D$. We put $f := [E : F]$, i.e.
$$E \cong L_f \subseteq L$$

and
$$F \subseteq E \subseteq B \subseteq A.$$

Firstly we need some lemmas. The actions of G on square lattice functions by conjugation induces maps
$$m_g : \ \Omega \to \Omega, \ x \mapsto g.x$$

and
$$c_g : \ \mathcal{I}_E \to \mathcal{I}_{gEg^{-1}}, \ y \in \mathrm{Latt}^2_{o_E}(B) \mapsto gyg^{-1} \in \mathrm{Latt}^2_{o_{gEg^{-1}}}(gBg^{-1})$$

for $g \in G$.

Lemma 11.5 $|m_g|$ and c_g induce isomorphisms on the simplicial structures of the Euclidean buildings, which preserve the orientation, i.e. an oriented edge is mapped to an oriented edge such that the direction is preserved. In particular $|m_g|$ and c_g are affine bijections, m_g preserves the embedding type, c_g the local type, and the following diagram is commutatve:

$$
\begin{array}{ccc}
\mathcal{I}^{E^\times} & \xrightarrow{\ |m_g|\ } & \mathcal{I}^{(gEg^{-1})^\times} \\
\downarrow{\scriptstyle j_E} & & \downarrow{\scriptstyle j_{gEg^{-1}}} \\
\mathcal{I}_E & \xrightarrow[\ c_g\ ]{} & \mathcal{I}_{gEg^{-1}}
\end{array}
$$

The following lemma gives a geometric interpretation of the map

$$\{\text{embedding types}\} \rightarrow \{\text{embedding types of vertices}\}$$

$$\langle \lambda \rangle \mapsto \langle \text{row}(\lambda)^T \rangle.$$

Lemma 11.6 (rank reduction lemma) Assume there is a field extension $K|F$ of degree s in $E|F$, where $2 \leq s \leq m$. Let $\mathfrak{a}$ be a vertex in $\Omega^{E^\times}$ such that $\mathfrak{a} \cap Z_A(K)$ is a face of rank s in $\Omega_K^{E^\times}$ and assume $(E, \mathfrak{a})$ has embedding type $\langle \lambda \rangle$ and $(E, \mathfrak{a} \cap Z_K(A))$ has embedding type $\langle \lambda' \rangle$. Then we get

$$\text{row}(\lambda) \sim \text{row}(\lambda'), \text{ i.e. } \lambda \sim \text{row}(\lambda')^\mathsf{T}.$$

Proof: By lemma 11.5 it is enough to show the result only for one embedding equivalent to $(E, \mathfrak{a})$. For simplicity we can restrict ourself to the case of $s = 2$. The argument for $s > 2$ is similar. We fix a D-basis of V. It is $(E, \mathfrak{a})$ equivalent to the pearl embedding $(E_\lambda, \mathfrak{a}_\lambda)$ of λ, moreover $\mathfrak{a}_\lambda$ is $M_m(o_D)$. Now we apply a permutation p on $(E_\lambda, \mathfrak{a}_\lambda)$ such that the odd exponents of σ in $pE_\lambda p^{-1}$ are behind all even exponents, i.e. $pE_\lambda p^{-1}$ is the image of

$$x \in L_f \mapsto \text{diag}(M_{n_1}(x), M_{n_2}(x)), \ n_1 := \sum_{i \text{ odd}} \lambda_i, \ n_2 := \sum_{i \text{ even}} \lambda_i$$

where

$$M_{n_1}(x) = \text{diag}(\sigma^0(x)\mathbb{1}_{\lambda_1}, \sigma^2(x)\mathbb{1}_{\lambda_3}, \ldots, \sigma^{f-2}(x)\mathbb{1}_{\lambda_{f-1}})$$

and

$$M_{n_2}(x) = \text{diag}(\sigma^1(x)\mathbb{1}_{\lambda_2}, \sigma^3(x)\mathbb{1}_{\lambda_4}, \ldots, \sigma^{f-1}(x)\mathbb{1}_{\lambda_f}).$$

For the embedding $(E', \mathfrak{a}')$ obtained by conjugating $p(E_\lambda, \mathfrak{a}_\lambda)p^{-1}$ with the matrix

$$\text{diag}(\mathbb{1}_{n_1}, \pi_D^{-1}\mathbb{1}_{n_2})$$

we have the following properties. Let $K'|F$ be the field extension of degree two in $E'|F$.

- K' is the image of the diagonal embedding of L_2 in $M_m(D)$ and its centraliser is $M_m(\Delta_{K'})$, where $\Delta_{K'} := Z_D(L_2)$. This follows because even powers of π_D commute with L_2.

- The intersection of $\mathfrak{a}'$ with $M_m(\Delta_{K'})$ is a herditary order in standard form with invariant $\langle n_1, n_2 \rangle$. The positivity of the integers n_i follows from the assumption that this intersection is a face of rank 2.

Since $\pi_{\Delta_{K'}} := \pi_D^2$ is a prime element of $\Delta_{K'}$ which normalises L and since the powers of σ occuring in the description of E' are even we can read the embedding type of

$(E', \mathfrak{a}' \cap \mathrm{M}_m(\Delta_{K'}))$ directly. It is the class of

$$
\begin{pmatrix}
\lambda_1 & \lambda_2 \\
\lambda_3 & \lambda_4 \\
\vdots & \vdots \\
\lambda_{f-1} & \lambda_f
\end{pmatrix}.
$$

Thus the result follows. q.e.d.

The next lemma shows that changing the skewfield does not change the embedding type.

Lemma 11.7 (changing skewfield lemma) Let D' be a central skewfield over a local field F' of index d with a maximal unramified extension field L' normalized by a prime element $\pi_{D'}$ and assume that V' is an m dimensional right vector space over D'. Denote the Euclidean building of $GL_m(D')$ by $\mathcal{I}'$ and let Σ, Σ' be an apartment of $\mathcal{I}$, $\mathcal{I}'$ corresponding to a basis (v_i), (v_i') respectively. Then Σ' is fixed by the image E' of the diagonal embedding of L_f' in $\mathrm{M}_m(D')$. Assume further that E is the image of the diagonal embedding of L_f in $\mathrm{M}_m(D)$. Under these assumptions the map $\equiv$ from $|\Sigma|$ to $|\Sigma'|$ defined by

$$
[x \mapsto \bigoplus_i v_i \mathfrak{p}_D^{[d(x+\alpha_i)]+}] \mapsto [x \mapsto \bigoplus_i v_i' \mathfrak{p}_{D'}^{[d(x+\alpha_i)]+}]
$$

is the geometric realisation from an isomorphism ϕ of simplicial complexes which preserves the orientation and the embedding type. The latter means that if $\mathfrak{a}'$ is the image of a hereditary order $\mathfrak{a}$ under ϕ then the embedding types of $(E, \mathfrak{a})$ and $(E', \mathfrak{a}')$ equal.

Proof: We define ϕ to map the class of a lattice chain $\mathfrak{L}$ with

$$
\mathfrak{L}_j = \bigoplus_i v_i \mathfrak{p}_D^{\nu_{i,j}}
$$

to the class of $\mathfrak{L}'$ with

$$
\mathfrak{L}_j' = \bigoplus_i v_i' \mathfrak{p}_{D'}^{\nu_{i,j}}.
$$

We only show the preserving of the embedding type. The other properties are verified easely. We take the two lattice chains $\mathfrak{L}$ and $\mathfrak{L}'$ with corresponding hereditary orders $\mathfrak{a}$ and $\mathfrak{a}'$. Applying from the left an appropriative permutation matrix P and an apropritative diagonal matrix T (resp. T'), whose entries are powers of the corresponding prime element, we obtain simultanously lattice chains corresponding to hereditary orders $\mathfrak{b}$, $\mathfrak{b}'$ in the same standard form. More precisely T' is obtained from T if π_D is substituted by $\pi_{D'}$. Thus $(TPEP^{-1}T^{-1}, \mathfrak{b})$ and $(T'PE'P^{-1}T'^{-1}, \mathfrak{b}')$ have the same embedding type and thus by conjugating back $(E, \mathfrak{a})$ and $(E', \mathfrak{a}')$ have the same embedding type. q.e.d.

We now fix a D-basis $v_1, \ldots, v_m$ of V and therefore a frame

$$\mathfrak{R} := \{R_i := v_i D | \ 1 \leq i \leq m\}$$

and an apartment $\Sigma = \mathrm{Her}(A)_{\mathfrak{R}}$ of Ω. The algebra A can be identified with $\mathrm{M}_m(D)$. By the affine bijection $|\Sigma| \cong \mathbb{R}^{m-1}$ which maps

$$[\Lambda] \text{ with } \Lambda(x) = \bigoplus_i \mathfrak{p}_D^{[d(x+\alpha_i)]+}$$

to

$$d(\alpha_1 - \alpha_2, \ldots, \alpha_{m-1} - \alpha_m),$$

we can introduce affine coordinates on $|\Sigma|$ where the points of $|\Sigma|$ corresponding to the vectors 0, $(f, 0, \ldots, 0)$, $(0, f, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, f)$ are denoted by $Q_1, Q_2, \ldots, Q_m$.

Remark 11.8 The vertices of Σ are exactly the points of

$$Q_1 + \sum_{i=2}^{m} \frac{1}{f}\mathbb{Z}(Q_i - Q_1).$$

Remark 11.9 For an element $g \in \cap_{i=1}^{m}(\mathrm{End}_D(R_i))^{\times}$, i.e. a diagonal matrix, $|m_g|$ induces an affine bijection of $|\Sigma|$. If g is $\mathrm{diag}(1, \ldots, 1, \pi_D^k, 1, \ldots, 1)$, with π_D^k in the i-th row, the map $|m_g|$ is of the form

$$Q \mapsto Q + \frac{k}{f}(Q_{i+1} - Q_i),$$

where we set $Q_{m+1} := Q_1$.

Example 11.10 Let us assume E is the image of the diagonal embedding of L_f in $\mathrm{M}_m(D)$, i.e.

$$E = \{(x, \ldots, x) | \ x \in L_f\}.$$

Then B and j_E simplify, i.e.

1. $B = \mathrm{End}_\Delta(W)$ with $\Delta := \mathrm{Z}_D(L_f)$ and $W := \bigoplus_i v_i \Delta$

2. The geometric realisation of Σ is a subset of $\mathcal{I}^{E^{\times}}$.

3. For $[\Lambda] \in \mathcal{I}$ we have

    $$j_E([\Lambda]) = [\Lambda \cap W]$$

    where $\Lambda \cap W$ denotes the lattice function

    $$x \mapsto \Lambda(x) \cap W.$$

11. Embedding types through barycentric coordinates

4. The image of $j_E|_{|\Sigma|}$ is the geometric realisation of the apartment Σ_E which belongs to the frame $\{v_i\Delta \mid 1 \le i \le m\}$ and in affine coordinates the map has the form

$$x \in \mathbb{R}^{m-1} \mapsto \frac{1}{f} x \in \mathbb{R}^{m-1}.$$

5. The vertices of Σ_E are the points of $|\Sigma_E|$ with affine coordinate vectors in $\mathbb{Z}^{m-1}$. Specifically the points $P_i := j_E(Q_i)$ are vertices of a chamber of Σ_E.

6. The edge from P_i to P_{i+1} is oriented to P_{i+1}.

Proof: [example] To prove the statements of the example it is enough to calculate j_E in terms of lattice functions, i.e to show 3. The statements then follow by similar and standard calculations.

For 2: We have $|\Sigma| \subseteq \mathcal{I}^{E^\times}$ because for an o_D-lattice function Λ split by $\mathfrak{R}$ the action of an element of $E^\times$ on Λ is the multiplication of every lattice $\Lambda(t)$ by a fixed element $x \in D^\times$.

For 3: We use the decomposition

$$V = W \otimes_\Delta D = W \oplus W\pi_D \oplus W\pi_D^2 \oplus \ldots \oplus W\pi_D^{f-1},$$

the function

$$[\Gamma] \in \mathcal{I}^{E^\times} \mapsto [\Lambda] \in \mathcal{I}$$

with

$$\Lambda(x) := \bigoplus_{i=0}^{f-1} \Gamma(x - \frac{i}{d})\pi_D^i$$

is affine and $B^\times$-equivariant. By 10.2 it has to be j_E^{-1} and thus

$$j_E([\Lambda]) = [\Lambda \cap W].$$

The appearence of j_E in terms of coordinates follows now from

$$\mathfrak{p}_D^{[x]+} \cap \Delta = \mathfrak{p}_\Delta^{[\frac{[x]+}{f}]+} = \mathfrak{p}_\Delta^{[\frac{x}{f}]+}.$$

q.e.d.

Proof: (of theorem 11.3) By lemma 11.5 and by theorem 8.13 we can assume that we are in the situation of the example 11.10 above and that there is a diagonal matrix h consisting of powers of π_D with exponents in $\mathbb{N}_{f-1} \cup \{0\}$ such that

$$(hEh^{-1}, h\mathfrak{a}h^{-1})$$

is the pearl embedding of λ. We consider two cases for the proof.

Case 1: $\mathfrak{a}$ has rank 1, i.e.
$$h\mathfrak{a}h^{-1} = \mathrm{M}_m(o_D) = Q_1$$

and λ is only one column. We get $\mathfrak{a}$ from Q_1 by applying $m_{h^{-1}}$ which is a composition of maps m_g where g differs from the identity matrix by only one diagonal entry π_D^k. Now remark 11.9 gives

$$\mathfrak{a} = Q_1 - \sum_{j=1}^{m} \frac{a_j}{f}(Q_{j+1} - Q_j)$$

where $a_j := k - 1$ if

$$\sum_{i=1}^{k-1} \lambda_i < j \leq \sum_{i=1}^{k} \lambda_i.$$

Thus in barycentric coordinates $j_E(M_{\mathfrak{a}})$ has the form

$$\frac{f - a_m + a_1}{f} P_1 + \frac{a_2 - a_1}{f} P_2 + \ldots + \frac{a_m - a_{m-1}}{f} P_m.$$

and therefore the vector

$$\mu := (\frac{f - a_m + a_1}{f}, \frac{a_2 - a_1}{f}, \ldots, \frac{a_m - a_{m-1}}{f})$$

fullfils part one of the theorem. If $(\lambda_{i_l})_{1 \leq l \leq s}$ is the subsequence of non-zero entries we define the indexes

$$j_l := \lambda_1 + \ldots + \lambda_{i_{l-1}} + 1$$

and $j_1 := 1$. This are the indexes where the μ_j are non-zero, more precisely from

$$j_l = \sum_{i=1}^{i_l - 1} \lambda_i + 1 \leq \sum_{i=1}^{i_l} \lambda_i$$

we obtain for a_j the following values:

$$a_j = a_{j_l} = i_l - 1, \ j_l \leq j < j_{l+1}$$

and

$$a_j = a_{j_s} = i_s - 1, \ j_s \leq j \leq m,$$

and thus the subsequence of non-zero entries of $f\mu$ is

$$(f\mu_{j_l}) = (f - i_s + i_1, i_2 - i_1, i_3 - i_2, \ldots, i_s - i_{s-1}).$$

Therefore we get for pairs($\langle f\mu \rangle$) the expression

$$\langle (f - i_s + i_1, \lambda_{i_1}), (i_2 - i_1, \lambda_{i_2}), (i_3 - i_2, \lambda_{i_3}), \ldots, (i_s - i_{s-1}, \lambda_{i_s}) \rangle$$

and this is precisely $\langle \mathrm{row}(\lambda) \rangle^c$.

Case 2: Assume the rank r of $\mathfrak{a}$ is not 1. Here we want to use rank reduction. We fix an unramified field extension $L'|F$ of degree rd in an algebraic closure of F. Denote by D' a skewfield which is a central cyclic algebra over F with maximal field L' and an

11. Embedding types through barycentric coordinates

L'-normalising prime element $\pi_{D'}$, i.e.

$$D' = \bigoplus_{i=0}^{dr-1} L'\pi_{D'}^i,$$

$$\pi_{D'}L'\pi_{D'}^{-1} = L', \text{ and } \pi_{D'}^{d+r} = \pi_F.$$

The images of L'_r, L'_{rf} under the diagonal embedding of L' in $\mathrm{M}_m(o_{D'})$ are denoted by F', E' respectively and the apartment of the Euclidean building Ω' of $GL_m(D')$ corresponding to the standard basis is denoted by Σ', i.e. we have a field tower

$$E' \supseteq F' \supseteq F$$

and apartments Σ', $\Sigma'_{E'}$, $\Sigma'_{F'}$ in the buildings $\mathcal{I}'$, $\mathcal{I}'_{E'}$, $\mathcal{I}'_{F'}$ respectively. We then obtain a commutative diagram of bijections, where the lines are induced by isomorphisms of chamber complexes which preserve the orientation.

$$
\begin{array}{ccc}
|\Sigma| & \xrightarrow{\ \equiv_F\ } & |\Sigma'_{F'}| \\
\downarrow{\scriptstyle j_E} & & \downarrow{\scriptstyle j_{E'}} \\
|\Sigma_E| & \xrightarrow{\ \equiv_E\ } & |\Sigma'_{E'}|
\end{array}
$$

The map $\equiv_F$ is given by

$$[x \mapsto \bigoplus_{i=0}^{m-1} v_i \mathfrak{p}_D^{[d(x+\alpha_i)]+}] \mapsto [x \mapsto \bigoplus_{i=0}^{m-1} e_i \mathfrak{p}_{Z_{D'}(L'_r)}^{[d(x+\alpha_i)]+}]$$

and $\equiv_E$ analogously. Here (e_i) is the standard basis of D'^m. Because of lemma 11.7 the map $\equiv_F$ preserves the embedding type and thus we can finish the proof by applying lemma 11.6 on

$$\Sigma' \rightarrow \Sigma'_{F'} \rightarrow \Sigma'_{E'}.$$

More precisely, let S_r be a face of rank r in $\Sigma'_{F'}$. Its barycenter has affine coordinates in $\frac{1}{r}\mathbb{Z}^{m-1}$ and therefore the preimage of it under $j_{F'}$ is a point S_1 with integer affine coeffitients, i.e. it corresponds to a vertex of $\mathcal{I}'$. To emphasise the base field we write field extensions as the index of j. Because of

$$j_{E'|F'}(M_{S_r}) = j_{E'|F'}(j_{F'|F}(S_1)) = j_{E'|F}(S_1)$$

the theorem follows now from the rank reduction lemma and case 1. q.e.d.

Part III.

Appendix, references and indexes

A. The Weil-restriction

Good references are [Wei82, 1.3.] and [KMRT98, 20.5]

Let $E|F$ be a finite separable field extension, and let V be an affine variety defined over E. The functor

$$B \mapsto V(E \otimes_F B)$$

from the category of commutative F-algebras to the category of sets is represented by an absolutely reduced finitely generated F-algebra $\tilde{A}$, see for example the proof in [KMRT98, 20.6]. The affine variety corresponding to $\tilde{A}$ is called *Weil-restriction* of V from E to F and denoted by $\mathrm{Res}_{E|F}(V)$.

Another way to construct the Weil-restriction is the following. One introduces coordinates in choosing an F-basis in E and the polynomial equations defining V become polynomial equations with coefficients in F. The set of solutions of these equations is the Weil-restriction of V from E to F and the map

$$e \in E \mapsto (\sigma(e))_\sigma$$

induces an isomorphism

$$\mathrm{Res}_{E|F}(V) \cong \prod_{\sigma:E \hookrightarrow \bar{F}} V^\sigma$$

defined over the normal hull of E. Here σ runs over the set of F-algebra monomorphisms from E into $\bar{F}$.

B. The building of a valuated root datum

The aim of this section is to give the definition of a valuated root datum and its building as it is done in [BT72, chapter 6 and 7]. For the theory of root systems see [Bou81, chapter IV].

Let V be a finite dimensional $\mathbb{R}$-vector space and let Φ be a root system in V^*. We denote its dual root system in V by $\hat{\Phi}$. The reflection r_a of V corresponding to $a \in \Phi$ is defined by

$$r_a(v) := v - a(v)\hat{a}.$$

The Weyl-group of Φ, i.e. the group generated by all r_a, $a \in \Phi$, is denoted by ${}^v W$. We take an invariant positive definite inner product on V and we get a canonical isomorphism from V to V^* via

$$v \mapsto (v, *).$$

It transfers the action of ${}^v W$ to V^*. The Weyl-group stabilizes Φ and $\hat{\Phi}$. The fixed point sets of the orthogonal reflections r_a give a cell decomposition of V, see for example [Bro89, chapter 1]. The chambers of V are in one to one correspondence to the bases of Φ. We fix a basis of Φ. Let Φ^+ be the set of positive roots of Φ corresponding to this basis.

B.1. Valuated root datum

Definition B.1 [BT72, 6.1.1] Let G be an arbitrary group. A system

$$(T, (U_a, M_a)_{a \in \Phi})$$

is a *root datum of type Φ in G* if the following holds.

- (DR 1) The sets T and U_a are subgroups of G. The groups U_a are non-trivial.

- (DR 2) For all roots a, b the commutator group $[U_a, U_b]$ is a subset of the group generated by all U_{na+mb} where n and m run over all natural numbers for which $na + mb$ is a root.

- (DR 3) If a and $2a$ are elements of Φ then U_{2a} is proper subset of U_a.

- (DR 4) The set M_a is a right coset of T in G and it holds

$$U_{-a}^* := U_{-a} \setminus \{1\} \subseteq U_a M_a U_a.$$

- (DR 5) For all roots a and b and all $m \in M_a$ on has $m U_b m^{-1} \subseteq U_{r_a(b)}$.

- (DR 6) The group U^+ generated by all U_a with positive root a and the group U^- generated by all the other U_a have the property that the intersection of $T.U^+$ with U^- is $\{1\}$.

A root datum of type Φ is *generative* if G is generated by the union of T and all U_a, $a \in \Phi$, i.e.

$$G = < T \cup \bigcup_{a \in \Phi} U_a > .$$

Remark B.2 [BT72, 6.1.2(4),(9),(10)] Given a root datum the cosets M_a, M_{-a} and M_a^{-1} equal and are determined by $(T, (U_a)_{a \in \Phi})$. Let N be the group generated by T and all M_a. There is a group epimorphism

$$^v\mu : \ N \to {}^vW$$

such that

$$nU_an^{-1} = U_{v\mu(n)}.$$

One has for example $\mu(M_a) = \{r_a\}$.

The initials DR stand for "donnée radicielle" the name given in [BT72]. Such a root datum can be defined for any reductive group defined over k.

Example B.3 [BT72, 6.1.3] Let k be a field.
Step 1: The group $SL_2(k)$ has a generative root datum of type A_1

$$(T, M_1, M_{-1}, U_1, U_{-1})$$

where T is the set of diagonal matrices, U_1 (resp. U_{-1}) the set of unitary upper (resp. lower) triangular matrices and M_1 the set of antidiagonal matrices in $SL_2(k)$.
Step 2: Let G be a split, affine, connected and simple group defined over k. We fix a maximal k-split torus T of G. Let Φ be the set of roots $\Phi(G,T)_k$ of the action of $T(k)$ on $\mathrm{Lie}(G)(k)$. By [Bor91, 18.7] there is a unique family of unipotent connected closed k-subgroups $(U_a)_{a \in \Phi}$ of G such that there are k-isomorphisms

$$\theta_a : \ \mathbf{A}^1(\bar{k}) \to U_a$$

satisfying

$$\mathrm{Inn}(t) \circ \theta_a(x) = \theta_a(a(t)x) \text{ for all } x \in \mathbf{A}^1(\bar{k}), \ t \in T.$$

One can choose the maps θ_a such that following two assertions hold.

- For every root a there is an isomorphism from $SL_2(\bar{k})$ to the subgroup generated by U_a and U_{-a} which maps the upper and the lower triangular unitary matrix with non-diagonal entry u to $\theta_a(u)$ and $\theta_{-a}(u)$ respectively.

- For every pair of roots a and b such that $a \neq -b$ there is a family of integers

$(C_{a,b,n,m})$ such that

$$[\theta_a(u), \theta_b(u')] = \prod_{n,m} \theta_{na+mb}(C_{a,b,n,m} u^n u'^m), \ u, \ u' \in \mathbf{A}^1(\bar{k})$$

where n and m run over the natural numbers with $na + mb \in \Phi$.

The system $(T(k), (U_a(k))_{a \in \Phi})$ is part of a generative root datum of type Φ in $G(K)$.

Step 3: Let G be a split semi-simple connected group defined over k. Then by [Bor91, 22.10] G is an almost direct product of the minimial closed connected normal k-subgroups G_i of G of strictly positive dimension. A maximal k-split torus T of G is the image of a maximal k-split torus $\prod T_i$ of $\prod_i G_i$ where T_i is a maximal k-split torus of G_i because the canonical isogeny

$$f : \prod_i G_i \rightarrow G$$

is central, see [Bor91, 22.9, 22.6]. f is also separable, i.e. the differential $d_e f$ is surjective, and therefore we have

$$f(\prod_i G_i(k)) = G(k)$$

The map $d_e f$ is in fact the isomorphism

$$\oplus_i \text{Lie}(G_i) \cong \text{Lie}(G)$$

and taking $\prod_i (T_i)$-fixed points on the left and T-fixed points on the right side we obtain

$$\oplus_i \text{Lie}(T_i) \cong \text{Lie}(T).$$

Thus

$$\prod_i T_i \rightarrow T$$

is separable and we obtain

$$f(\prod_i T_i(k)) = T(k).$$

We now take for every i a root datum

$$(T_i(k), (M_a(k), U_a(k))_{a \in \Phi(G_i, T_i)})$$

as done in G_i by step 2. We now apply f on the product of the root data and we obtain a generative root datum

$$(T(k), (M_a(k), U_a(k))_{a \in \Phi(G,T)})$$

of type $\Phi(G, T)$ in $G(k)$.

Step 4: We assume now that G is an affine reductive group defined over k. Then the group $G^0 / \text{Rad}(G)$ is affine, connected, semisimple and defined over k by [Bor91, 18.2(ii),Prop. 11.21, 6.8]. Thus we can assume that G is semisimple and connected. Let

T be a maximal k-split torus of G. One can choose a maximal torus T' of G which is defined over k and contains T by the remark below. T' is split over a finite separable extention k' of k. We take $\Phi' := \Phi(G, T')$ and the groups U_a', $a \in \Phi'$, obtained from step 2. For $a \in \Phi := \Phi(G, T)_k$ we define U_a as the closed subgroup of G generated by all $U_{a'}$ where a is the restriction of a' to T. The tupel

$$(\mathrm{Z}_G(T)(k), (U_a(k))_{a \in \Phi})$$

is part of a generative root datum of type Φ of $G(k)$.

Remark B.4 If G is a connected reductive k-group and T is a maximal k-split torus of G. We can choose a maximal torus of G containing T which is defined over k.

Proof: A maximal torus S containing T is split over k^{sep} and is a subset of $H := \mathrm{Z}_G(T)^0$, which is normalized by S and k-closed. Thus by [Bor91, 20.3] H is defined over k^{sep}. The separability of $k^{sep}|k$ implies that H is defined over k. The theorem [Bor91, 18.2(i)] ensures the existence of a maximal torus of H which is defined over k. This torus contains T and it is a maximal torus of G because it is conjugated to S in H. q.e.d.

We now come to the definition of a valuation of a root datum.

Assumption B.5 *Let*
$$RD := (T, (U_a, M_a)_{a \in \Phi})$$

be a generative root datum of type Φ of a group G. We put $U_{2a} := \{1\}$ if $a \in \Phi$ and $2a \notin \Phi$.

Definition B.6 [BT72, 6.2.1] A *valuation* of RD is a family $\phi = (\phi_a)_{a \in \Phi}$ of maps

$$\phi_a : \ U_a \to \mathbb{R} \cup \{\infty\}$$

such that the following contitions hold.

1. (V0) For every a the image of ϕ_a has at least 3 elements.

2. (V1) For every a and for every $r \in \mathbb{R} \cup \{\infty\}$ the set

$$U_{a,r} := \phi_a^{-1}([r, \infty])$$

   is a subgroup of U_a and $U_{a,\infty}$ is trivial. For the images one writes

$$\Gamma_a := \phi_a(U_a^*) \text{ and } \Gamma_a' := \{\phi_a(u) \mid u \in U_a^*, \phi_a(u) = \sup \phi_a(uU_{2a})\}.$$

3. (V2) For every a and for every $m \in M_a$, the function

$$u \mapsto \phi_{-a}(u) - \phi_a(mum^{-1})$$

   is constant on U_{-a}^*.

4. (V3) For $a, b \in \Phi$ with $b \notin -\mathbb{R}_+ a$ and $k, l \in \mathbb{R}$ the commutator group $[U_{a,k}, U_{b,l}]$ is contained in the group generated by $U_{pa+qb,pk+ql}$, $p, q \in \mathbb{N}$ with $pa + qb \in \Phi$.

5. (V4) If a and $2a$ are in Φ the map ϕ_{2a} is the restriction of $2\phi_a$ on U_{2a}.

6. (V5) If $a \in \Phi$, $u \in U_a$ and $u', u'' \in U_{-a}$ such that $u'uu'' \in M_a$ then

$$\phi_a(u) = -\phi_{-a}(u').$$

Remark B.7 One has $\Gamma_a' = \Gamma_a$ if $2a$ is not in Φ.

Definition B.8 A valuation ϕ is *discret* if every group Γ_a is a discret subgroup of $\mathbb{R}$.

If (k, ν) is a non-Archimedian local field there is a valuation of the root datum of B.3 by [BT72, 6.2.3 and chapter 10]. For $\mathrm{SL}_2(k)$ it is given by

$$\phi_1\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) := \phi_{-1}\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) := \nu(u).$$

and in the case of a split, semisimple k-group by

$$\phi_a(\theta_a(t)) := \nu(t).$$

These valuations and the valuations considered in part 1 of this thesis are discret.

Remark B.9 Let ϕ be a valuation of RD and let $\lambda: \Phi \to \mathbb{R}_+^*$ be a function which is constant on the irreducible components of Φ. For a vector $v \in V$ the family $\psi := \lambda\phi + v$ defined by

$$u \mapsto \lambda(a)\phi_a(u) + a(v)$$

is again a valuation of the root datum. If λ is the constant function 1, then one says that ψ is a *translation* of ϕ by v.

Definition B.10 [BT72, 6.2.5] Two valuations are *equipollent* if the second is a translation of the first by a vector of V. The group N acts on the set of valuations of RD in the following way. If w is the element ${}^v\mu(n)$ for some $n \in N$ one puts

$$(n.\phi)_a(u) := \phi_{w^{-1}(a)}(n^{-1}un).$$

For $n \in N$, $v \in V$ and $\lambda: \Phi \to \mathbb{R}$ on has

$$n.(\lambda\phi + v) = \lambda(n.\phi) + {}^v\nu(n)(v).$$

B.2. Building of a valuated root datum

[BT72, chapter 7]

Assumption B.11 *In addition to assumption B.5 we fix a valuation ϕ of RD.*

Let Δ be the set of valuations of RD equipollent to ϕ. It is an affine space over V and we identify Δ and V in choosing ϕ as the zero of Δ. The action of N on the set of valuations of RD restricts to an action of Δ and it defines a map ν from N to the group of affine automorphisms of Δ. Its kernel is denoted by H. The set of affine roots of Φ in Δ is the collection of the closed halfspaces

$$\alpha_{a,k} := \{x \in \mathcal{A} \mid a(x) + k \geq 0\}, a \in \Phi, k \in \Gamma'_a.$$

The set of affine roots is denoted by Σ. We put $U_{\alpha_{a,k}} := U_{a,k}$ For a non-empty subset S of Δ one defines

- U_S to be the group generated by the U_α where α runs over the affine roots containing S, and

- $P_S := HU_S$.

The *Bruhat-Tits building $\mathcal{F}$ of G with respect to RD and ϕ* is the set of equivalence classes of $G \times \Delta$ under the relation

$$(g,x) \sim (h,y) \text{ if and only if there exists an } n \in N$$

such that

$$y = \nu(n)(x) \text{ and } g^{-1}hn \in P_{\{x\}}.$$

It is a G-set under the action on the first coordinate. A subset Δ' of $\mathcal{F}$ is said to be an *apartment* of $\mathcal{F}$ if there is an element g of G such that $\Delta' = g\Delta$.

This definition does not need that ϕ is discrete. For the description of the faces we assume that ϕ is discrete, for the description in the general case see [BT72, 7.2]. The faces of Δ are the cells of the cell decomposition given by the hyperplanes which are boundaries of affine roots, see [Bro89, chapter 1]. A subset S of $\mathcal{F}$ is a *face* if there is a face S' of Δ and an element g of G such that $S' = gS$.

C. The enlarged building of a reductive group

Assumption C.1 *In this section k is a non-Archimedean local field with residual characteristic different from 2.*

We follow the explaination in [BT84a, 4.2.16] The buildings introduced in [Tit79] are already enlarged.

Let G be a connected affine reductive group defined over k together with a Bruhat-Tits building $\mathcal{F}$. Let $X^*(G)_k$ be the group of characters of G defined over k and let V^1 be the dual $\mathbb{R}$-vector space of $X^*(G)_k \otimes_{\mathbb{Z}} \mathbb{R}$. The *enlaged affine building of* $(G, \mathcal{F})$ is the set $\mathcal{F} \times V^1$ equipped with the $G(k)$-action

$$g.(x, v) := (g.x, v + \theta(g))$$

where $\theta(g)$ is defined by

$$\theta(g)(\chi) := -\nu(\chi(g)).$$

Apartments and faces carry over from $\mathcal{F}$ to $\mathcal{F}^1$ in the natural way.

Definition C.2 We say that there exists a proper enlarged building over k if $X^*(G)_k$ is not trivial.

We now discuss the cases where an enlarged building occurs in the case of the classical groups considered in this thesis. We fix a hermitian k-datum

$$((A, V, D), \rho, k_0, h, \epsilon, \sigma)$$

and we analyse below when $X^*(\mathbf{SU}(h))_{k_0}$ or $X^*(\mathbf{U}(h)^0)_{k_0}$ are trivial.

Theorem C.3 *[Bor91, Cor. 14.2] Let G be an affine reductive group. Then the following conditions are equivalent.*

1. *The group is semisimple, i.e. the maximal normal connected solvable subgroup $\mathcal{R}(G)$ is trivial.*

2. *The connected component equals its commutator subgroup.*

3. *The center of G^0 is finite.*

Remark C.4 A semisimple connected group equals to its commutator subgroup which implies the triviality of the character group. Examples for semisimple connected groups

are $\mathbf{SL}_n(\bar{k})$, $\mathbf{Sp}_{2n}(\bar{k})$, and $\mathbf{SO}_{n'}(\bar{k})$ for $n, n' \geq 1$ except $n' = 2$. The connectivity is seen using transvections and the semisimplicity follows because these groups are generated by the images of connected subsets of $\mathbf{SL}_2(\bar{k})$. By proposition 1.34 we obtain a trivial character group for $\mathbf{SU}(h)$ if σ is

- symplectic, or

- orthogonal and $md \neq 2$, or

- unitary.

We firstly analyse the unitary case.

Lemma C.5 The group $X^*(\mathbf{U}(h))_{k_0}$ is trivial if σ is unitary.

Proof: We have $D = k$ by theorem 1.37. Using the isomorphism

$$(\operatorname{End}_k(V) \otimes_{k_0} \bar{k}, \sigma \otimes_{k_0} \bar{k}) \cong (\mathrm{M}_m(\bar{k}) \times \mathrm{M}_m(\bar{k}), \tilde{\sigma})$$

with

$$\tilde{\sigma}(B, C) = (C^T, B^T)$$

we obtain for a k_0-rational character χ of $\mathbf{U}(h)$ that its restriction to $U(h)$ must be a power of the determinant. The involution σ is conjugated to the transpostion which implies

$$\chi(x) = \chi(\sigma(x))$$

for all $x \in U(h)$. In addition $\sigma(x)$ is the inverse of x for $x \in U(h)$. Thus the only possible values of χ on $U(h)$ are 1 and -1. Thus χ is trivial because $\mathbf{U}(h)$ is connected and $U(h)$ is Zariski-dense in $\mathbf{U}(h)$ by [Bor91, 18.3]. q.e.d.

Lemma C.6 Let σ be orthogonal and assume $dm = 2$. There exist a proper enlarged Bruhat-Tits-building for $\mathbf{SU}(h)$ over k if and only if $d = 1$ and h is isotropic.

Before we start the proof we recall that the *k-rank* of a reductive connected k-group is the dimension of a maximal k-split torus.

Proof: We have $\mathbf{SU}(h) \cong \mathbf{G_m}(\bar{k})$ defined over k if d is one and h is isotropic, i.e. all characters are k-rational and the character group is free of rank one.

If $d = 2$ there is an isomorphism from $\mathbf{SU}(h)$ to $\mathbf{G_m}(\bar{k})$ defined over $\bar{k}$ but not over k because of the different k-ranks. There is an element $a \in SU(h) \setminus \{1, -1\}$ because $SU(h)$ is Zariski-dense in $\mathbf{SU}(h)$ by [Bor91, 18.3]. The degree of D over k is 2 and therefore the centralizer of $k[a]$ in D is $k[a]$, especially the commutative group $SU(h)$ is a subset of $k[a]$. In addition $k[a]$ is invariant under σ. Thus we can apply lemma C.5 and we obtain that there is no polynomial multiplicative map from $SU(h)$ to $\mathbf{G_m}(k)$.

For the last part of the proof we assume that $d = 1$ and $\mathbf{SU}(h)$ is anisotropic. There is a k-basis of V such that the Gram-matrix of h is of the form

$$\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$$

and we identify A with $M_2(k)$. A short calculation shows that

$$\mathbf{SU}(h) = \left\{ \begin{pmatrix} a & cf \\ -ce & a \end{pmatrix} \mid a, c \in \bar{k} \text{ s.t. } a^2 + efc^2 = 1 \right\}.$$

We fix square roots $\sqrt{e}$ and $\sqrt{-f}$. The conjugation with

$$\begin{pmatrix} \sqrt{e} & \sqrt{-f} \\ \frac{\sqrt{e}}{2} & -\frac{\sqrt{-f}}{2} \end{pmatrix}$$

maps $\mathbf{SU}(h)$ to $\mathrm{SU}(\tilde{h})$ where the Gram-matrix of $\tilde{h}$ under the standart basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The explicit formula for the map is

$$\begin{pmatrix} a & cf \\ -ce & a \end{pmatrix} \mapsto \begin{pmatrix} a - c\sqrt{-ef} & 0 \\ 0 & a + c\sqrt{-ef} \end{pmatrix}.$$

Thus a character of $\mathbf{SU}(h)$ is of the form

$$\begin{pmatrix} a & cf \\ -ce & a \end{pmatrix} \mapsto (a + c\sqrt{-ef})^z$$

for some integer z. The inverse of $(a + c\sqrt{-ef})$ is $(a - c\sqrt{-ef})$. If z is positive in the binomial expansion of $(a + c\sqrt{-ef})$ the coeffitient in front of $\sqrt{-ef}$ is zero because $\sqrt{-ef} \notin k$ because h is anisotropic. Thus a k-rational character χ of $\mathbf{SU}(h)$ fulfils

$$\chi(x) = \chi(x^{-1})$$

for all $x \in \mathrm{SU}(h)$. The density of $\mathrm{SU}(h)$ in $\mathbf{SU}(h)$ and the connectivity of $\mathbf{SU}(h)$ imply that χ is trivial. q.e.d.

If we summarize the two lemmas and the remark we obtain the following proposition.

Proposition C.7 $X^*(\mathbf{SU}(h))_{k_0} \neq 1$ *if and only if* $m = 2$ *and* $d = 1$ *and* σ *is orthogonal and* h *is isotropic. If* σ *is unitary there is no nontrivial k-rational character of* $\mathbf{U}(h)$.

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